PARAMETER ESTIMATION IN NON-LINEAR ROTOR-BEARING SYSTEMS FROM RANDOM RESPONSE

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

by

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to the

DEPARTMENT OF MECHANICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

OCTOBER, 1995

26/10/95

CERTIFICATE

It is certified that the work contained in the thesis entitled, "**PARAMETER ESTIMATION IN NON-LINEAR ROTOR-BEARING SYSTEMS FROM RANDOM RESPONSE**" has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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Thesis Title: PARAMETER ESTIMATION IN NON-LINEAR ROTOR-BEARING SYSTEMS FROM RANDOM RESPONSE

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During the last two decades, use of statistical models and methods for the analysis of dynamic phenomena has become widespread in various fields of natural sciences and technology. Statistical dynamics, concerned with the study of various random phenomena in dynamic systems, enriches the classical basic theory of oscillations. It also extends the possibilities for its application to the description and analysis of real response processes in dynamic systems, which in many cases may be of quite irregular or random nature. In fact, many phenomena can be described adequately only within the frame work of statistical models i.e. deterministic models prove to be inadequate, or at least extremely complex. Problems of this kind arise, frequently, in rotor dynamics, especially those involving random vibrations of bearings (rolling element and fluid film both), excited by random loads. Moreover, rotor bearing problems are inherently strongly nonlinear (Dimentberg, 1988; Bendat, 1990; Soong and Grigoriu, 1993; Lin and Cai, 1995; Ragulskis et al, 1974; Harris, 1984; Stolarski, 1990; Zhou and Hashimoto, 1995; Choy et al, 1992; Childs, 1993).

Rolling element bearing are known to possess highly non-linear elastic characteristics and the complexities and approximate nature of the analytical determination of these properties are responsible for some of the unreliability in the prediction of the response and stability of a rotor system. Procedures [Ragulskis, et. al., 1974; Harris, 1984] are available for estimation of bearing stiffness under static loading conditions in *isolation of shaft*. The existing parameter estimation techniques for overall *rotor-bearing systems* (e.g. Muszynska and Bently, 1990) involve, *a controlled input excitation* , that needs to be given to the bearings.

The present study attempts for the development of a parameter estimation procedure, for the non-linear elastic parameters of bearings, that can be used *on-line*, and is based on the analysis of easily accessible vibration signals, picked up from the bearing caps.

The forces setting the rotor-bearing system comprise of both - harmonic forces, due to rotor unbalance and random forces due to bearing surface imperfections, caused by random deviations from their standard theoretical design and progressive surface and subsurface deterioration. In addition excitation can be contributed by from random inaccuracies in the rotor-bearing-housing assembly etc. Stochastical dynamics methods provide an opening, to attempt the inverse problem of parameter estimation, in such a rotor-bearing system, governed by a non-linear equation of motion, having both deterministic and random excitation. Under certain engineering assumptions, the random excitation can be taken to be an ideal white noise process and the response of a dynamic system (linear or non-linear) can be modeled as a diffusive Markov Process or an approximate Markov Process.

The structure of a Markov Process is completely determined for all future times by the distribution at some initial time and by a transition probability density function, which satisfies the Fokker-Planck-Kolmogorov (FPK) equation. The FPK equation can be solved to obtain analytical expressions for the first order probability distribution of the stationary response. For the response to be modeled by a Markov Process, it is necessary that the excitations be approximated by ideal white noises. This restriction can be, in principle, removed at the price of increasing the complexity of the system. Linear filters are introduced between the ideal white-noise excitations and the system, to produce realistic excitation spectra for the nonlinear system. The ideal white-noise excitations, driving the filters, guarantee that the response of the extended system is a Markov Process (of higher order), for which an extended FPK equation can be written. For such an approximate Markovian modeling stochastic averaging is applied, whereby rapid fluctuations are averaged to provide simpler equations for slowly fluctuating quantities. The procedure is a nontrivial extension of the Krylov-Bogoliubov (1937) averaging method for deterministic excitations, since it involves accounting for the averaged effect of a random excitation multiplied by a correlated response.

The problem is initially attempted for *balanced* rotors, by considering the random forces from the bearings as the primary source of excitation, which are generally large enough to cause measurable level of vibrations. The dynamics of the rotor-bearing system is modeled as a Markov Process and Fokker-Planck equations are formulated. The drift and diffusion coefficients in the FPK equation are derived from the nonlinear equations of motion of the dynamic system (Crandall, 1958). The approach to the solution of equation is greatly simplified if the overall random excitation to the system, from the variety of sources, is treated as ideal white noise. While many engineering applications are based on this idealization, insofar as the excitation itself is concerned, it turns out that the response obtained through such models are quite acceptable if the time scale

of excitation is much smaller than the time scale of the response (Lin, 1973). The time scale for the excitation is the correlation time, roughly defined as the length of time separation beyond which the excitation process is nearly uncorrelated. The time scale of the response is the measure of the memory duration of the system which is generally about one quarter of the natural period of a mode which contributes significantly to the total response.

The Fokker-Planck equations for the rotor-bearing system are solved, to obtain the first order distribution of the response, which is further processed for the inverse problem of parameter estimation. The linear and nonlinear bearing stiffness parameters are obtained. The technique has a distinct advantage that, it does not require an estimate of the excitation forces and works directly on the response signals from the bearing caps. In this analysis the rotor is treated as a rigid body. The algorithm is tested by Monte Carlo simulation. The procedure is illustrated for a laboratory rotor rig and the results are compared with those from the analytical formulations of Harris (1984) and Ragulskis et al. (1974).

The analysis becomes more involved, if the shaft flexibility is taken into account. The present study further attempts the problem of bearing stiffness parameter estimation for flexible rotors. In contrast to the rigid rotor case, which could be treated as a single degree freedom problem, the flexible rotor poses a nonlinear multi-degree of freedom problem. The excitation to the balanced system is taken to be random in nature, primarily arising out of bearing and assembly imperfections. The problem is formulated for a flexible shaft carrying a centrally located rigid disc and supported at the ends by nonlinear bearings. The inverse problem is approached by initially effecting a coordinate transformation, so as to enable the governing equations to be modeled as Markov Processes through the Fokker-Planck equations. The solution to the Fokker-Planck equation is obtained under certain engineering assumptions. A curve fitting algorithm is developed to process the statistical response of the system obtained by the solution of the Fokker-Planck equation, to extract the rotor-bearing stiffness parameters.

The problem, for rotors with flexible shafts and carrying more than one disc, is considered next. The governing nonlinear differential equations for such multi-mass flexible systems are derived for balanced rotors with random excitation at the bearings. The governing equations are subjected to a coordinate transformation and modeled as Markov Processes. General form expressions, for the first order probability statistics of the response are obtained. The statistical response is processed to extract the rotor-bearing stiffness parameters. The procedure is illustrated for a laboratory test-rig with two discs and the

experimental results are compared with the analytical guidelines of Harris (1984) and Ragulskis et al. (1974). The algorithm developed is tested by Monte Carlo numerical simulation procedure.

The above studies concern single-degree-of-freedom and multi-degree-of-freedom systems, where the rotor is assumed to be balanced. The study, further, explores the possibility of estimation of linear and non-linear stiffness parameters, for rotors with unknown unbalance. The problem is formulated as single degree of freedom system. The case of a rotor with a rigid shaft, in nonlinear flexible bearings is investigated. The excitation to the system consists harmonic forces due to the unbalance and random forces due to arbitrary deviations, of bearing contact surfaces and subsurfaces, from their ideal design and their progressive deterioration during operation. These random forces are comparable to the harmonic excitation forces, if the unbalance is not significantly large. The parameter estimation procedure is based on the averaging technique of Bogoliubov and Mitropolsky (1961) for deterministic non-linear systems, extended by Stratonovich (1967) for stochastic differential equations. The governing equation of motion is transformed from the rapidly varying variables, namely displacement and velocity, to variables, amplitude and phase, varying slowly with time. Stochastic averaging is done, to take into account the effect of the random excitation multiplied by a correlated term, so as to model the slowly varying amplitude as an approximate Markovian process. A second order stochastic approximation is carried out and a one-dimensional Fokker-Planck equation is derived to describe the Markovian amplitude process. The response to the Fokker-Planck equation is derived and processed for parameter estimation. Along with the bearing stiffness parameters, estimates of the unknown unbalance of the rotor, its angular location and the damping ratio are also obtained, as by-products. The procedure is illustrated for a laboratory rotor rig and the experimental results are validated through comparisons with the available analytical guidelines.

To summarize, non-linear bearing stiffness estimation procedure, based on statistical methods, have been developed for cases of rigid rotors, single disc flexible rotors, multi-disc flexible rotors. The study is extended, to include harmonic excitation to the non-linear system along with random excitation and the case of an unbalanced rigid rotor is discussed. The estimation procedure has an advantage over existing ones, for it does not require an estimate of the excitation forces and works directly on the measured response signals of the system and therefore can be employed on-line. The algorithms are illustrated for a laboratory rotor-bearing test rig and the results are compared with those obtained through an existing analytical model.

ACKNOWLEDGMENTS

I feel fortunate to have worked with Dr. N. S. Vyas. He was source of encouragement, direction and help, all along my association with him. I am extremely thankful to him for the tremendous amount of support he provided to me, at every stage of my thesis.

I thank Dr. B. Ravindra, Mr. S. Chatterji, Mr. N. Shekhar, Mr. Selvam and Mr. G. K. Singh for useful discussions with them during the course of this work and to Mr. M. M. Singh for his timely help.

I cherished the homely atmosphere created by members of family of Prof. A. K. Mallik, Prof. V. Sundararajan, Dr. N. S. Vyas and Mr. M. M. Singh. I am deeply indebted to my wife Vibha who has shown immense understanding and extented full cooperation throughout the last one and half years.

Finally, I am grateful to Structures Panel of the Aeronautical Research and Development Board of India, for providing financial support in carrying out this work.

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SYMBOLS

$A(t)$	*Amplitude*
A_{nf}	*Non-'fluctuating amplitude term*
A_{no}	*Non-'fluctuating amplitude*
$A_{nf\text{-}no}$	*Non-'fluctuating - non-'oscillatory amplitude term*
$A^{*}(t)$	*Non-'fluctuating amplitude*
c	*Damping factor*
c_1 , c_2	*Normalization constants*
e	*Eccentricity in disc*
$f(t)$	*Random excitation force per unit mass*
F	*Elastic force*
F_b	*Bearing force*
F_0	*Disc unbalance*
F_s	*Shaft stiffness force*
g	*Preload*
$g(x)$, $G(x)$	*Non-linear function*
k_L ,k_{NL}	*Linear and non-linear stiffness parameters of bearings*
k_{ij}	*Shaft stiffness parameter*
$k(x)$	*Bearing nonlinear stiffness*
K_n	*Coefficient of proportionality*
m	*Rotor mass*
m_1 , m_2	*Effective bearing masses*
m_i	*Disc masses*
M_κ	*Minors of the matrix* $[\kappa]$
N	*Number of sample points*
$p(x)$	*Probability density of displacement*
u, v	*Fluctuating part of the amplitude*
u_{ij} , v_{ij}	*Eigen vector elements*
x-y-z	*Rectangular coordinate system*
$x, \dot{x}, \ddot{x}$	*Displacement, velocity and acceleration, respectively*

S_0	*Uniform spectral density*
$S(\zeta(t), \omega)$	*Spectral density of random excitation $\zeta(t)$ at frequency ω*
t	*Time*
$V(x)$	*Potential energy term*
α_{ij}	*Shaft damping parameter*
β, β_{ij}	*Damping factor*
δ	*Direc delta function*
ΔA	*Amplitude increment*
Δt	*Time increment*
Δx	*Displacement increment*
ε	*Small parameter*
η_i	*Generalized coordinates*
ϕ_i	*Random excitation intensity factor*
$\varphi(t)$	*Phase angle*
φ_{nf}	*Non-'fluctuating Phase term*
φ_{no}	*Non-'fluctuating Phase term*
$\varphi_{nf\text{-}no}$	*Non-'fluctuating - non-'oscillatory Phase term*
$\varphi^*(t)$	*Non-'fluctuating Phase term*
γ	*Ratio (= $\beta/m\phi$)*
λ	*Nonlinear stiffness contribution parameter*
μ	*Disc mass to effective bearing mass ratio*
ξ	*Damping ratio*
θ	*Reference angle*
$\psi(t)$	*Random excitation force*
$\zeta(t)$	*Random excitation force*

CHAPTER 1

INTRODUCTION

Bearing analysis constitutes a major area in rotordynamic studies. The response and stability of a rotor system are critically dependent on bearing characteristics. It has become evident in recent years, that an important class of rotor bearing phenomenon cannot be studied without adequately accounting for the nonlinear forces produced by the bearings. Bearing nonlinearity assumes a greater role for high speed and low weight rotor applications, where the vibration amplitudes tend to be relatively large. While, dynamic characterization of fluid film bearings has drawn considerable attention from researchers, the information available, on the nonlinear aspects of rolling element bearings under dynamic conditions, is relatively scarce.

Rolling element bearings, and in particular ball bearings, despite their mechanical simplicity, are known to display highly nonlinear behaviour and present some very complex rotor problems. The approximate nature of the analytical determination of these bearing properties is responsible for some of the unreliability in the prediction of the response and stability of a rotor system. Procedures (Ragulskis, et al., 1974; Harris, 1984) are available for estimation of bearing stiffness under static loading conditions in *isolation of shaft*. The existing parameter estimation techniques for overall *rotor-bearing systems* (e.g. Muszynska and Bently, 1990) involve, *a controlled input excitation* , that needs to be given to the bearings.

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The forces setting the rotor-bearing system into vibrations comprise of both - harmonic forces, due to rotor unbalance and random forces due to bearing surface imperfections, caused by random deviations from their standard theoretical design and progressive surface and subsurface deterioration. In addition excitation can be contributed by from random inaccuracies in the rotor-bearing-housing assembly etc.

Stochastical dynamics methods provide an opportunity, to attempt the inverse problem of parameter estimation, in such a rotor-bearing system, governed by a non-linear equation of motion and having both

deterministic and random excitations. Under certain engineering assumptions, the random excitation can be taken to be an ideal white noise process and the response of a dynamic system (linear or non-linear) can be modeled as a diffusive Markov Process or an approximate Markov Process.

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The study, further, explores the possibility of estimation of linear and non-linear stiffness parameters, for rotors with unknown unbalance. The problem is formulated as single degree of freedom system. The case of a rotor with a rigid shaft, in nonlinear flexible bearings is investigated. The excitation to the system consists

harmonic forces due to the unbalance and random forces due to arbitrary deviations, of bearing contact surfaces and subsurfaces, from their ideal design and their progressive deterioration during operation. These random forces are comparable to the harmonic excitation forces, if the unbalance is not significantly large. The parameter estimation procedure is based on the averaging technique of Bogoliubov and Mitropolsky (1961) for deterministic non-linear systems, extended by Stratonovich (1967) for stochastic differential equations. The governing equation of motion is transformed from the rapidly varying variables, namely displacement and velocity, to variables, amplitude and phase, varying slowly with time. Stochastic averaging is done, to take into account the effect of the random excitation multiplied by a correlated term, so as to model the slowly varying amplitude as an approximate Markovian process. A second order stochastic approximation is carried out and a one-dimensional Fokker-Planck equation is derived to describe the Markovian amplitude process. The response to the Fokker-Planck equation is derived and processed for parameter estimation. Along with the bearing stiffness parameters, estimates of the unknown unbalance of the rotor, its angular location and the damping ratio are also obtained, as by-products. The procedure is illustrated for a laboratory rotor rig and the experimental results are validated through comparisons with the available analytical guidelines.

The drift and diffusion coefficients in the FPK equation can be derived from the nonlinear equations of motion of the dynamic system (Crandall, 1966). A general, closed form solution to FPK equation is yet to be found. When the time derivative in FPK equation is set equal to zero, the equation describes the first order probability distribution for the stationary response. The distribution is readily obtained for a limited class of vibratory systems with nonlinear restoring forces and special forms of nonlinear damping (Fuller, 1969; Caughey, 1971; Caughey and Ma, 1983; Pradlwarter et al., 1991). The Fokker-Planck equation was derived and the exact stationary response was obtained for certain cases of two-degree of freedom non-linear dynamic systems, by Ariaratnam (1960). The theory was generalized later by Caughey (1963) for multi-degree of freedom cases.

The exact mean-square response and average, frequently provided by the first-order distribution, have been widely used as a basis for comparison with results of various approximate methods and/or procedures for computer simulation. The time dependent solution to FPK equation i. e., the transition probability density function, has been obtained for only a few nonlinear systems (Caughey, 1971; Roberts, 1981a). Yong and Lin (1987). Lin and Cai (1988a,b) have developed a systematic procedure to obtain the exact stationary response for either external (additive) or parametric (multiplicative) excitations, or both. Exact stationary solution in terms of probability density function, for a class of non-linear systems driven by a non-normal-delta-correlated process has been obtained recently by Vasta (1995).

2.2 STOCHASTIC AVERAGING OR APPROXIMATE MARKOV PROCESS APPROACH

For the response to be approximated by a Markov process, it is necessary that the excitation be approximated by ideal white noise. This restriction can be, in principle, removed at the price of increasing the complexity of the system, by introducing linear filters between the ideal white-noise excitations and the system. The filters serve to produce realistic excitation spectra for the nonlinear system, while the ideal white-noise excitations driving the filters guarantee that the response of the extended system is a Markov process (of higher order), for which an extended FPK equation can be written. This strategy has often been suggested but it has, apparently, never been implemented for nonlinear response problems. However, it can be shown by using stochastic averaging principles that, under certain conditions, the response of a nonlinear dynamic system to non-white excitation can be approximated by a Markov process. The designation - stochastic averaging - has been applied to a class of procedures in which rapid fluctuations are averaged to provide simpler equations for slowly fluctuating quantities. The procedure is a nontrivial extension of the

Krylov-Bogoliubov (1937) averaging method for deterministic excitations, since it involves accounting for the averaged effect of a random excitation multiplied by a correlated response.

There are three distinct approaches to stochastic averaging. The first approach was introduced by Stratonovitch (1967) and rigorously justified by Khasminskii (1966) and by Papanicolaou and Kohler (1974). Approximate equations for the slowly varying quantities are obtained by time-averaging the rapid fluctuations. This method, called the standard stochastic averaging method (or Stratonovitch's stochastic averaging method), is applied to narrow-band responses which can be represented as sinusoidal oscillations with slowly varying amplitude and phase. The formulae of the stochastic averaging methods were also derived by Lin (1986) using a different procedure, making the physical implication of the method clearer and more appealing to engineers. A generalization of stochastic averaging in random excitation with special emphasis was placed on casting the problem in a more formal mathematical framework by Red-Horse and Spanos (1992). Zhu et. al. (1994) improved the stochastic averaging procedure. Bouc (1994) has obtained the random response with a higher order approximation in the presence of a large non-linear stiffness term in a dynamic system subjected to random excitations.

Another approach involves averaging the drift and diffusion coefficients in the FPK equation, with respect to time. The mathematical basis for this method is contained in a theorem by Khasminskii (1963) and is called the Averaging Method of Coefficients in FPK equation. One more approach is based on dividing the response variables into rapidly varying quantities and slowly varying quantities and approximate equations for the latter are obtained by averaging the rapid fluctuations of the former. The basis for this is also due to Khasminskii (1968). For a single degree of freedom oscillator, the rapidly varying quantity is the displacement and the slowly varying quantity is the energy envelope (Dimentberg, 1980; Roberts, 1983; Spanos, 1983; and Zhu, 1983). This method is called the Generalized Stochastic Averaging Method (or Stochastic Averaging Methods of Energy Envelope). Moshchuk et al. (1995) gave an analytical method based on the stochastic averaging of the energy envelope to treat the dynamic behaviour of single-degree-of-freedom elastic ocean structures subjected to non-linear hydrodynamic loading by Gaussian ocean waves.

All the above three approaches have been extensively applied to nonlinear random vibration of mechanical and structural systems as effective approximations for prediction of the response, stability and reliability. Among the three stochastic averaging methods, the Standard Stochastic Averaging Method finds the most popular application on account of its accuracy, applicability and simplicity (Zhu, 1988). For nonwhite

excitation in non-linear random dynamic system, Cai (1995) developed a procedure of quasi-conservative averaging. He solved the Duffing oscillator under both external and parametric excitations with nonwhite spectral densities.

2.3 STATISTICAL LINEARIZATION METHOD

A natural method of approaching a nonlinear problem is to replace the given set of nonlinear equations by an equivalent set of linear ones, since a linear system is so much easier to analyse. This is achieved by minimizing the difference between the non-linear and linear sets of equations in an appropriate sense. The basic development of a suitable linearization procedure (variously known as statistical linearization, equivalent linearization or stochastic linearization), for randomly excited nonlinear systems, is usually attributed to Booton et al (1953), Booton (1954) and Caughey (1963a). They generalized the deterministic linearization methods of Krylov and Bogoliubov (1937) to the stochastic case.

Although, rationalizations for the procedure usually appeal to a small nonlinearity argument, it fortuitously turns out that the error in mean-square response remains quite modest even for large nonlinearities (Jazwinski 1970). A great advantage of the statistical linearization method is that it can easily be generalized to cope with multi-degree of freedom systems, including those where hysteretic elements are incorporated. The earliest extensions of the theory in this direction were given by Caughey (1963a) and Kazakov (1965a , 1965b). Subsequently there have been a number of theoretical advantages in this area (Roberts 1981b). The technique has no difficulty in dealing with non-white excitations and can be further generalized to cope with non-stationary excitations and responses (Spanos 1981a, Roberts 1981b; Roberts and Spanos, 1990). Chang and Young (1989) obtained the stationary response of robot manipulators under stochastic base and external excitations by statistical linearization method. Socha and Soong (1991) reviewed and gave an assessment of the procedures in the area of statistical and equivalent linearization in the analysis of non-linear stochastic systems. Stochastic linearization on nonlinear multi-degree-of-freedom systems under random parametric excitation was examined by Falsone (1992). Elishakoff and Cai (1993) obtained an approximate solution for nonlinear random vibration problems by a partial stochastic linearization. Stochastic equivalent linearization was used by Chang (1994) to develop a finite element formulation for the dynamic response analysis of non-linear hysteretic plates subjected to random excitations. Soize (1995) described the calculation of the power

spectral density function of stationary response and the method of identification, of single degree of freedom non-linear second order dynamic systems excited by a white or a broad-band Gaussian noise. Grigoriu (1995) has developed an equivalent linearization method for the analysis of the non-linear systems with Poisson white noise input.

2.4 PERTURBATION METHOD

Perturbation methods are generally employed, when the amount of nonlinearity in a system is controlled by a small scaling parameter. The solution is sought in the terms of a power series in the small scaling parameter and successive terms are evaluated as linear responses to nonlinear functions of the preceding terms. This classical approach for deterministic nonlinear problems (Stoker, 1950) was extended to random vibration problems by Crandall (1963). In practice, the calculations are seldom carried beyond the first perturbation. The perturbation series is often an asymptotic series, so that the higher order perturbations improve the approximation for small scaling parameter at the expense of worsening it for large scaling parameter.

Functional series methods offer an alternative approach to developing an expansion, based on the linear solution. An example of the application of a such method is the work of Orabi and Ahmadi (1987). They used a Wiener-Hermite expansion and presented a formal procedure for deriving the deterministic equations governing the kernel functions, arising in the expansion (Roy and Spanos, 1990). A common difficulty with all expansion methods lies in establishing the regimes of convergence, in the appropriate parameter space. It is frequently found that, due to a combination of poor convergence properties and excessive computational requirements, it is only possible to obtain reliable results if there is a very small degree of nonlinearity.

2.5 METHOD OF MOMENTS

A set of differential equations for various statistical moments, or related quantities known as cumulants (or semi-invariants) and quasi-moments (Stratonovitch, 1967) of the response, as the function of time, can be obtained by multiplying the FPK equation by suitable functions and integrating over the probability space. Equivalent sets of equations can be derived directly from the dynamic equations of motions or the equivalent Itô equations. The response problem is thus reduced to a set of coupled ordinary differential equations. For linear systems the set of equations closes, in the sense that for some finite 'n' there is a set of 'n' independent equations in which only 'n' unknown moments appear. For nonlinear systems, the set of equations forms an

infinite hierarchy: There are always more unknown moments than equations. Approximate solutions have been proposed, based on ad-hoc closure assumption: Certain higher order moments are neglected or assumed to be related to lower order moments, in the same way that moments of Gaussian processes are related (Bolotin, 1979; Ibrahim and Roberts, 1978; Roberts, 1981a).

A non-Gaussian closure method, based on constructing an approximate representation of the system's probability density function with 'n' adjustable parameters and using 'n' independent moment equations to determine the parameters, was proposed by Dashevskii and Liptser (1967). A characteristic feature of this approach is that the complexity of the moment equations dramatically increases as the order of closures increases. However, results for nonlinear oscillators have been obtained for 'n' values up to 6, using both cumulant closure (Wu and Lin, 1984; Lin and Wu, 1984) and quasi-moment closure (Bover, 1978). These studies demonstrate that a significant improvement in accuracy can be obtained by progressing beyond simple Gaussian closure. The method has been applied by Ibrahim and coworkers (Ibrahim, 1985; Ibrahim and Soundararajan, 1985; Ibrahim et al, 1985) to a study of the response of systems with nonlinear inertial terms to random excitation. The method gives good results in some cases but is apparently not a very robust procedure (Crandall, 1985).

A second-order closure method was presented by Nayfeh and Serhan (1990) for determining the response of non-linear systems to combined deterministic and random excitations. A closure method was proposed by Grigoriu (1991) for calculating moments of the state vector of non-linear system satisfying an Itô stochastic differential equation. Paola et al. (1992) presented the extension of the Itô rule for the case of vector real functions of the response of nonlinear systems excited by zero-mean Gaussian white noise processes. Davies and Liu (1992) have obtained the response of a non-linear oscillator excited by random narrow-band noise by stochastic averaging followed by partial linearization. Ibrahim et al (1993) have examined stochastic bifurcation in moments of a clamped-clamped beam response to wide band random excitations analytically (Gaussian and non-Gaussian closure), numerically (Monte Carlo simulation) and experimentally. An efficient moment calculation method has been developed, recently, by Katafygiotis and Beck (1995).

2.6 METHOD OF EQUIVALENT NON-LINEAR EQUATIONS

An alternative generalization of statistical linearization has been proposed by Caughey (1986). The idea is to replace the original set of nonlinear differential equations by an equivalent nonlinear set, where the latter

belong to a class of problems which can be solved exactly. This class is, at present, very limited, and thus the range of applicability of the technique is correspondingly restricted. The results have been obtained for oscillators with nonlinearity in damping and stiffness (Caughey, 1986; Cai and Lin, 1988; Zhu and Yu, 1989). The equivalent statistical quadratization methods have been applied to non-linear multi-degree-of-freedom systems subjected to random excitation by Spanos and Donley (1991, 1992). It has been demonstrated that the method is very effective as a means of predicting the probability distribution of the response, with reasonable accuracy.

2.7 METHOD OF COMPUTER SIMULATION

Numerical simulation or the Monte Carlo method (Shinozuka, 1972; Bolotin, 1979; Spanos, 1981b; Rubinstein, 1981; Spanos and Mignolet, 1989) consists of generating a large number 'm' of sample excitations, computing the corresponding response samples and processing them to obtain the desired response statistics. The backbone, of any digital simulation study, is an algorithm which provides a set of pseudo-random numbers, belonging to a population with a specified probability density function. Proper processing of this set of numbers can yield the values of sample functions of random process excitations, with pre-selected frequency content and temporal variation of intensity, at successive discrete equi-spaced times. Upon generating a single sample of excitation, the commonly available subroutines for the numerical integration of the differential equations can be employed to obtain the system response. Another sample of the excitation can then be generated and the computed values of the system response can be used to update its statistics. The procedure is, in principle, very general and applicable to stationary or non-stationary response of systems of any degree of complication.

The statistical uncertainty in the response statistics decreases in proportion to 'm' while the computational cost increases essentially in proportion to 'm'. To gain one additional significant figure in a result requires a hundred fold increase in computational cost. Spanos (1981a) has estimated that, for cases where both statistical linearization and numerical simulation can be applied, the computational efficiency of the former will be of the order of 100 to 1000 times better than the latter. In applications where no exact solution is available numerical simulation is often used as a basis for assessing the accuracy of other approximate methods. Benaroya and Rehak (1988) reviewed the field of probabilistic structural analysis where finite element methods are used. The stochastic finite element analysis was applied to obtain response of geometrically and materially non-linear plane trusses under random excitations by Cherng and Wen (1991).

The finite element method is applied, in conjunction with the method of equivalent linearization, to large - deflection random response of thermally buckled beams by Locke and Mei (1990) and to the non-linear random response of beams by Locke (1994).

2.8 ROLLING ELEMENT BEARING VIBRATIONS AND STIFFNESS ESTIMATION

Estimation of the elastic parameters of bearings involves establishing a relationship between the incident load on the bearing and its resultant deformation. The classical solution for the local stress and deformation of two elastic bodies apparently contacting at a single point was established by Hertz in (1896). Hertz's analysis is applied to surface stresses caused by a concentrated force, applied perpendicular to the surface. In the determination of contact deformation versus load, the concentrated load applied normal to the surface alone, is considered, for most rolling element bearing applications. Methods of calculation of the surface and subsurface stresses of the combination of normal and tangential (traction) stresses are complex, (Zwirlein and Schlicht, 1980). Owing to infinitesimally small irregularities in the basic surface geometries of the rolling contact bodies, neither uniform normal stress field nor a uniform shear field are likely to occur in practice (Sayler et al, 1981; Kalker, 1982). Rigorous mathematical/numerical methods have been developed to calculate the distribution and magnitude of surface stresses in any line contact situation, that is, including the effects of crowning of rollers, raceways, and combinations thereof (Kunert, 1961; Reusner, 1977). Additionally, finite element methods (FEM) have been employed (Fredriksson, 1980) to perform the same analysis.

It is possible to determine how the bearing load is distributed among the balls or rollers, after having determined how each ball or roller in a bearing carries load. To do this it is necessary to develop load-deflection relationships for rolling elements contacting raceways. Most rolling bearing applications involve steady-state rotation of either the inner or outer raceways or both. Rolling element centrifugal forces, gyroscopic moments and frictional forces and moments do not significantly influence this load distribution in most applications. Theoretical models (Palmgren, 1959; Ragulskis et al., 1974; Harris, 1984; Eschamann et al. 1985; Stolarski, 1990) are available for estimation of bearing stiffnesses under static loading conditions.

Bearings concern vibrations caused due to geometric imperfections of contact surfaces were first analyzed by Lohman (1953) and Gustavsson (1962). Significant contributions have been made by Gupta et al (1975, 1977, 1978a, 1978b, 1978c, 1978d) towards the understanding of rolling element bearing dynamics.

Comprehensive investigations have been carried out on the high frequency response of bearings (McFadden and Smith, 1984) and its relation to surface irregularities (Sunnesjo, 1985; McFadden and Smith, 1985; Su et al, 1993). Lim and Singh have analyzed the vibration transmission through rolling element bearings in series of publications (1990a, 1990b, 1991, 1992, 1994).

A method for determination of the nonlinear characteristics of bearings using the procedure of Krylov-Bogoliubov-Mitropolsky has been suggested by Kononenko and Plakhtienko (1970). Honrath (1960) and Elsermans et al (1975) examined the stiffness and damping of rolling element bearing experimentally. Walford and Stone (1980) designed and fabricated a test rig for direct measurement of the relative displacement of the shaft and bearing housing for the oscillating force applied to the bearing housing, which is used to obtain the stiffness parameters. They found that the interfaces play between races and housing and shaft play a significant role in the determination of bearing stiffness and damping. Beatty and Rowan (1982) determined the dynamic stiffness of ball bearing. Kraus et al (1987) presented a method for the extraction of rolling element bearing stiffness and damping under operating conditions. The method is based on experimental modal analysis combined with a mathematical model of the rotor-bearing-support system. Effects of speed, preload and free outer race bearings on stiffness and damping have been investigated. The technique involves a controlled input excitation to be given to the bearings. Mitchell et. al (1966) obtained the stiffness of an oil film bearing, experimentally, by application of static loads. Morton (1971) devised the measurement procedure for estimation of the dynamic characteristics of a large sleeve bearing by application of dynamic loads using vibrators. Nordmann and Schollhorn (1980) identified the stiffness and damping coefficients of journal bearings by means of the impact method. Sahinkaya and Burrows (1984) estimated the linearised oil film parameters from the out-of-balance response where the shaft was excited by a known unbalance force. Muszynska (1990) has developed a perturbation technique for estimation of these parameters. Goodwin (1991) reviewed the experimental approaches to rotor support impedance measurement, with particular emphasis on fluid film bearing impedance measurement. A general procedure for identification of restoring force nonlinearity from a system's response to a white-noise excitation has been discussed by Dimentberg and Sokolov (1991). Nonlinear stochastic contact vibrations and friction at a Hertzian contact have been studied by Hess et al (1992). Analytical and experimental studies are carried out by them, using the Fokker-Planck equation and simulating the vibrations to the contact region, either externally by a white Gaussian random normal load or internally by a rough surface input. Childs and Hale (1993) devised a test apparatus and facility to identify the rotordynamic coefficients of high speed hydrostatic bearings.

CHAPTER 3

BEARING STIFFNESS ESTIMATION IN SINGLE-DISC RIGID ROTORS

The forces setting the rotor-bearing system into vibrations, comprise of both - harmonic forces, due to rotor unbalance and random forces due to bearing surface imperfections, caused by random deviations from their standard theoretical design and progressive surface and subsurface deterioration. The problem is initially attempted for balanced rotors, by considering the random forces from the bearings as the primary source of excitation, which are generally large enough to cause measurable level of vibrations.

3.1 EQUATION OF MOTION

The governing equation for a balanced rigid rotor supported at ends in bearings (Figure 3.1) with nonlinear stiffnesses can be written as

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2[x + \lambda G(x)] = f(t) \tag{3.1}$$

where G can be a polynomial in x and λ is the unknown non-linear stiffness contribution parameter. $f(t)$ in equation (3.1) represents the random excitation to the system caused due to bearing surface imperfections and progressive wear and tear. In addition excitation can be contributed by from random inaccuracies in the rotor-bearing-housing assembly etc.

The Markov vector approach is adopted for obtaining expressions for the response of the nonlinear random system of equation (3.1). The approach to the solution of equation (3.1) is greatly simplified if the overall random excitation to the system, from the variety of sources, is treated as ideal white noise. For an ideal white noise process, the response of a dynamic system (linear or non-linear) is a diffusive Markov process. The structure of a Markov process is completely determined for all future times by the probability distribution at some initial time and by a transition probability distribution, which satisfies the Fokker-Planck equation. While many engineering applications are based on this idealization, insofar as the excitation itself is concerned, it turns out that the response obtained through such models are quite

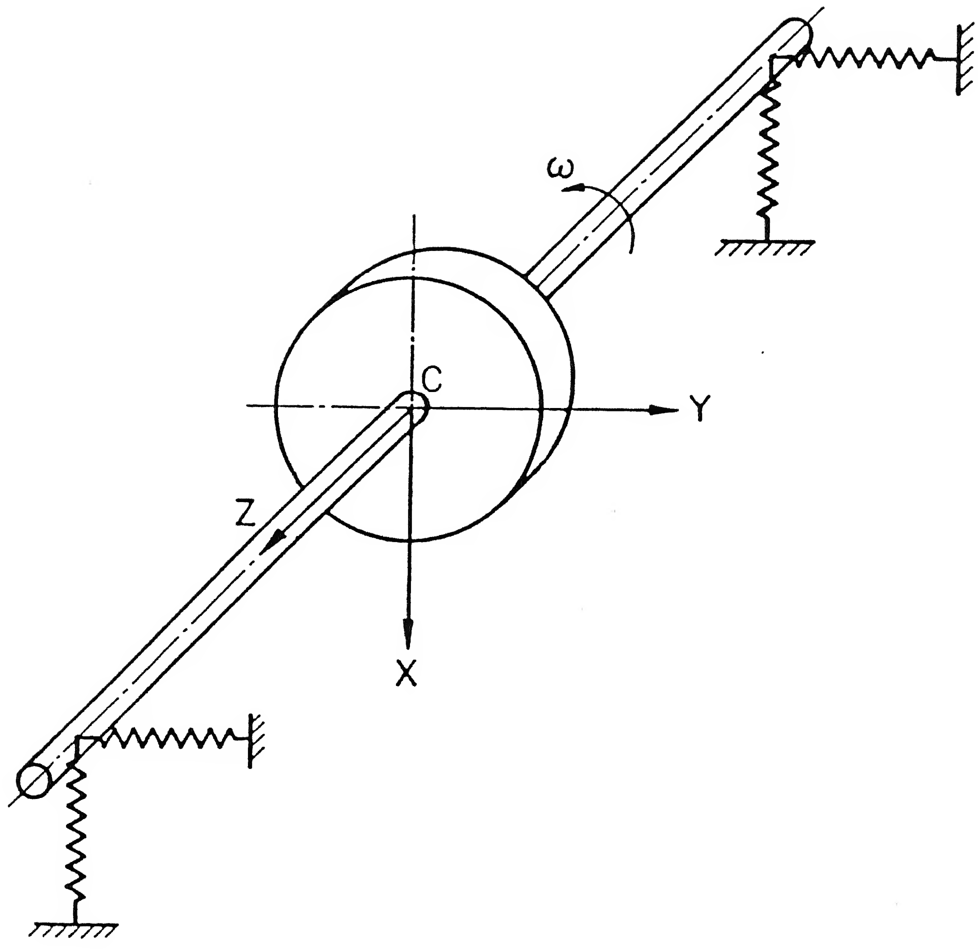


Figure 3.1 **Rotor on rolling element bearings**

acceptable if the time scale of excitation is much smaller than the time scale of the response (Lin, 1973). The time scale for the excitation is the correlation time, roughly defined as the length of time separation beyond which the excitation process is nearly uncorrelated. The time scale of the response is the measure of the memory duration of the system which is generally about one quarter of the natural period of a mode which contributes significantly to the total response. The statistical averages of the ideal white noise can be expressed as

$$
\begin{aligned}
&< f(t_1) >= 0 \\
&< f(t_1) f(t_2) >= 2\pi S_o \delta(t_2 - t_1)
\end{aligned}
\tag{3.2}
$$

where S_0 is the uniform spectral density of excitation and δ denotes Direc delta function.

3.2 F-P-K EQUATION

For a set of state space equations

$$
\begin{aligned}
&\partial x_i / \partial t = \dot{x}_i \\
&\partial \dot{x}_i / \partial t = f_i(t) - (1/M_i)(\beta_{ii}\dot{x}_i - \partial V(x_1, x_2, \ldots, x_N)/\partial x_i)
\end{aligned}
\qquad i = 1, 2, \ldots, N
\tag{3.3}
$$

with $f_i(t)$, being the zero mean white noise excitation, as defined in equations (3.2), the response is a diffusive Markov process. for which the transition probability density function is governed by the Fokker-Planck equation, given as (Caughey, 1963)

$$
\begin{aligned}
\partial p / \partial t = &-\sum_{i=1}^{N} [\partial / \partial x_i \{a_i p\} + \partial / \partial \dot{x}_i \{b_i p\}] \\
&+ (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} [\partial^2 / \partial \dot{x}_i^2 \{c_{ii} p\} + \partial^2 / \partial \dot{x}_j^2 \{d_{jj} p\} + \partial^2 / \partial \dot{x}_i \partial \dot{x}_j \{e_{ij} p\}]
\end{aligned}
\tag{3.4}
$$

where N is the number of degrees of freedom in the dynamic-system-model.

The first order moments, a_i , b_i , in the Fokker-Planck equation, called the drift coefficients, are given by

$$a_i(x,t) = \lim_{\Delta t \to 0} (< \Delta x_i > / \Delta t)$$
$$b_i(\dot{x},t) = \lim_{\Delta t \to 0} (< \Delta \dot{x}_j > / \Delta t) \qquad i = 1, 2, \ldots ,N \qquad (3.5)$$

and the second order moments, c_{ii} , d_{jj} , e_{ij} , called the diffusion coefficients, are expressed as

$$c_{ii}(x,t) = \lim_{\Delta t \to 0} (< \Delta x_i^2 > / \Delta t)$$
$$d_{jj}(\dot{x},t) = \lim_{\Delta t \to 0} (< \Delta \dot{x}_j^2 > / \Delta t) \qquad i = 1, 2, \ldots ,N \qquad (3.6)$$
$$e_{ij}(x,\dot{x},t) = \lim_{\Delta t \to 0} (< \Delta x_i \Delta \dot{x}_j > / \Delta t)$$

The governing equation, (3.1), for the rotor-bearing system under consideration can be rewritten as

$$dx / dt = \dot{x}$$
$$d\dot{x} / dt = f(t) - 2\zeta\omega_n \dot{x} - \omega_n^2 x - \omega_n^2 \lambda G(x) \qquad (3.7)$$

and its drift and diffusion coefficients, from equations (3.5) and (3.6), obtained as

$$a_1 = \dot{x}$$
$$b_1 = -2\zeta\omega_n \dot{x} - \omega_n^2 x - \omega_n^2 \lambda G(x) \qquad (3.8)$$

and

$$c_{11} = 0$$
$$d_{11} = 2\pi S_0 \qquad (3.9)$$
$$e_{11} = 0$$

The Fokker-Planck equation, for the rotor-bearing system, can then be written as

$$-\dot{x}\frac{\partial p}{\partial x} + \frac{\partial}{\partial \dot{x}}(2\varsigma\omega_n \dot{x} p) + \omega_n^2 [x + \lambda G(x)]\frac{\partial p}{\partial \dot{x}} + \pi S_0 \frac{\partial^2 p}{\partial \dot{x}^2} = \frac{\partial p}{\partial t} \qquad (3.10)$$

3.3 RESPONSE

For a stationary case, equation (3.10) reduces to

$$-\dot{x}\frac{\partial p}{\partial x}+\frac{\partial}{\partial \dot{x}}(2\varsigma\omega_n\dot{x}p)+\omega_n^2[x+\lambda G(x)]\frac{\partial p}{\partial \dot{x}}+\pi S_0\frac{\partial^2 p}{\partial \dot{x}^2}=0 \tag{3.11}$$

for which the solution is

$$p(x,\dot{x})=c\ exp\ [-\frac{2\varsigma\omega_n}{\pi S_0}\{\frac{1}{2}\dot{x}^2+\frac{1}{2}x^2\omega_n^2+\lambda\omega_n^2 g(x)\}]$$

where

$$g(x)=\int_0^x G(\xi)d\xi \tag{3.12}$$

The probability density functions $p(x)$ and $p(\dot{x})$ are obtained from equation (3.12) as (Roberts and Spanos, 1990), as

$$p(x)=\int_{-\infty}^{\infty} p(x,\dot{x})d\dot{x}$$

$$=c_1\ exp\ [-\frac{2\varsigma\omega_n^3}{\pi S_0}\{\frac{1}{2}x^2+\lambda g(x)\}]$$

where

$$c_1=1/\int_{-\infty}^{\infty}\ exp\ [-\frac{2\varsigma\omega_n^3}{\pi S_0}\{\frac{1}{2}x^2+\lambda g(x)\}]dx \tag{3.13}$$

and

$$p(\dot{x})=\int_{-\infty}^{\infty} p(x,\dot{x})dx$$

$$=\frac{1}{\pi}\sqrt{\frac{\varsigma\omega_n}{S_0}}\ exp\ [-\frac{2\varsigma\omega_n}{\pi S_0}\{\frac{1}{2}\dot{x}^2\}] \tag{3.14}$$

The variance of the velocity response is

$$\sigma_{\dot{x}}^2 = \int_{-\infty}^{\infty} \dot{x}^2 p(\dot{x}) d\dot{x} \\ = \frac{\pi S_0}{2\varsigma\omega_n} \tag{3.15}$$

From equations (3.13) and (3.15), the probability density function for the displacement response can be written as

$$p(x) = c_1 \exp\left[-\frac{\omega_n^2}{\sigma_{\dot{x}}^2}\left\{\frac{1}{2}x^2 + \lambda g(x)\right\}\right]$$

with

$$c_1 = 1 \Big/ \int_{-\infty}^{\infty} \exp\left[-\frac{\omega_n^2}{\sigma_{\dot{x}}^2}\left\{\frac{1}{2}x^2 + \lambda g(x)\right\}\right] dx \tag{3.16}$$

3.4 EXTRACTION OF BEARING STIFFNESS PARAMETERS

The bearing stiffness parameters are obtained from the experimentally obtained random response in terms of the linear parameter ω_n^2 and the nonlinear stiffness parameter λ. These parameters are obtained for both, the vertical and horizontal directions, the problem formulation, in the horizontal direction, remaining identical to that in the vertical direction. (The stiffness cross-coupling in rolling element bearings being negligible, the governing equations in the vertical and horizontal directions are taken to be independent, during the entire course of the present study.)

The probability density function for any two displacements x_i and x_{i+1} $(x_{i+1} > x_i)$, from equation (3.16) are

$$p(x_i) = c_1 \exp\left[-\frac{\omega_n^2}{\sigma_{\dot{x}}^2}\left\{\frac{1}{2}x_i^2 + \lambda g(x_i)\right\}\right] \tag{3.17}$$

and

$$p(x_{i+1}) = c_1 \; exp \; [-\frac{\omega_n^2}{\sigma_{\dot{x}}^2}\{\frac{1}{2}x_{i+1}^2 + \lambda g(x_{i+1})\}] \tag{3.18}$$

Defining

$$\Delta x_i = x_{i+1} - x_i \tag{3.19}$$

so that

$$\begin{aligned} x_{i+1}^2 &= (x_i + \Delta x_i)^2 \\ &\approx x_i^2 + 2x_i \Delta x_i \end{aligned} \tag{3.20}$$

for small Δx_i

and

$$\begin{aligned} g(x_{i+1}) &= g(x_i + \Delta x_i) \\ &\approx g(x_i) + \frac{dg(x)}{dx}|_{x=x_i} \Delta x_i \end{aligned} \tag{3.21}$$

and substituting from equations (3.20) and (3.21) into equation (3.18), gives

$$p(x_{i+1}) = c_1 \; exp \; [-\frac{\omega_n^2}{\sigma_{\dot{x}}^2}\{\frac{1}{2}x_i^2 + \lambda g(x_i)] \; exp \; [-\frac{\omega_n^2}{\sigma_{\dot{x}}^2}\{x_i + \lambda \frac{dg(x)}{dx}|_{x=x_i}\}\Delta x_i] \tag{3.22}$$

Combining equations (3.22) and (3.17) gives

$$p(x_{i+1}) = p(x_i) \; exp \; [-\frac{\omega_n^2}{\sigma_{\dot{x}}^2}\{x_i + \lambda \frac{dg(x)}{dx}|_{x=x_i}\}\Delta x_i] \tag{3.23}$$

For N displacement values, $x_1, x_2, \ldots\ldots\ldots, x_N$, equation (3.23) can be expressed as a set of N-1 linear simultaneous algebraic equations, as,

$$\frac{\sigma_{\dot{x}}^2}{\Delta x_i} \; ln \; [\frac{p(x_i)]}{p(x_{i+1})}](\frac{1}{\omega_n^2}) - \frac{dg(x)}{dx}|_{x=x_i} \lambda = x_i \tag{3.24}$$

$$i = 1, 2 \ldots\ldots\ldots\ldots N-1$$

Equations (3.24) are solved for $1/\omega_n^2$ and λ, using the least square fit technique. The variance, $\sigma_{\dot{x}}^2$ and the probability function, $p(x)$, are computed from the experimentally obtained displacement and velocity data (x and $\dot{x}$), which are taken as zero mean Gaussian processes and the nonlinear spring force provided by the rolling element bearings is taken to be cubic in nature i.e. $G(x) = x^3$, based on the work of Ragulskis et al (1974). They conducted exhaustive experimental work on isolated ball bearing and found that cubic nonlinearity best fits their experimental data. Subsequently several other researchers have also employed the cubic nonlinearity model.

3.5 EXPERIMENTATION

The laboratory rig for the experimental illustration of the technique is shown in Figures 3.2, 3.3 and 3.4. The rig consists of a disc centrally mounted on a shaft supported in two identical ball bearings. The shaft is driven through a flexible coupling by a motor and the vibration signals are picked up (after balancing the rotor, balancing is done at 1000 rpm and vibration signals are picked up at that speed only) in both, the vertical and horizontal directions, by accelerometers mounted on one of the bearing housings. The signals from the accelerometers are digitized on a PC/AT (Figure 3.5) after magnification. (The specifications of the instrumentation is given in the Appendix - A)

Typical displacement and velocity signals, in the vertical direction, picked up by the accelerometer are given in Figures 3.6 and 3.7. The probability density function, $p(x)$, of the displacement is shown in Figure 3.8. The bearing parameters estimated from equations (3.24) are -

Vertical direction:

$\omega_n^2 = 5.42 \times 10^7$ (rads/sec)^2

$\lambda = -1.27 \times 10^6$ (mm^{-2})

Horizontal direction:

$\omega_n^2 = 3.21 \times 10^7$ (rads/sec)^2

$\lambda = -1.29 \times 10^6$ (mm^{-2})

(Refer to Appendix C for statistical Error Table)

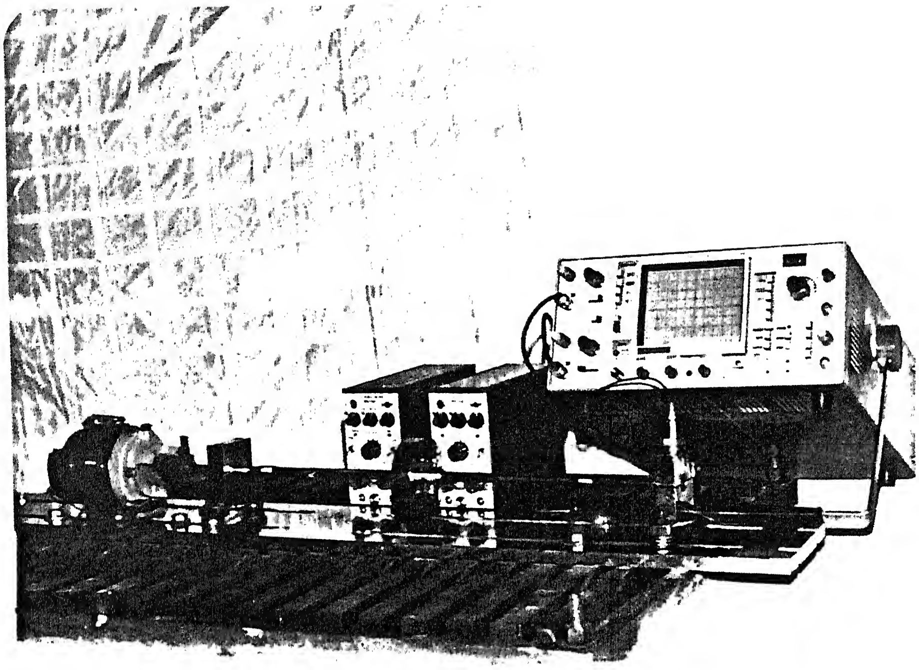


Figure 3.2 Laboratory rotor bearing rig

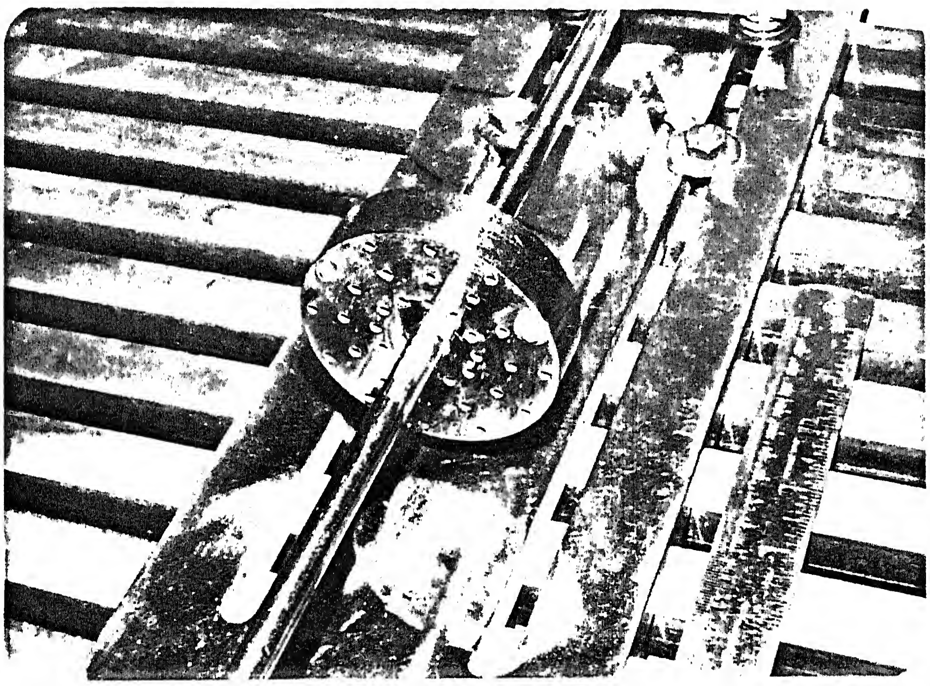


Figure 3.3 Close view of disc

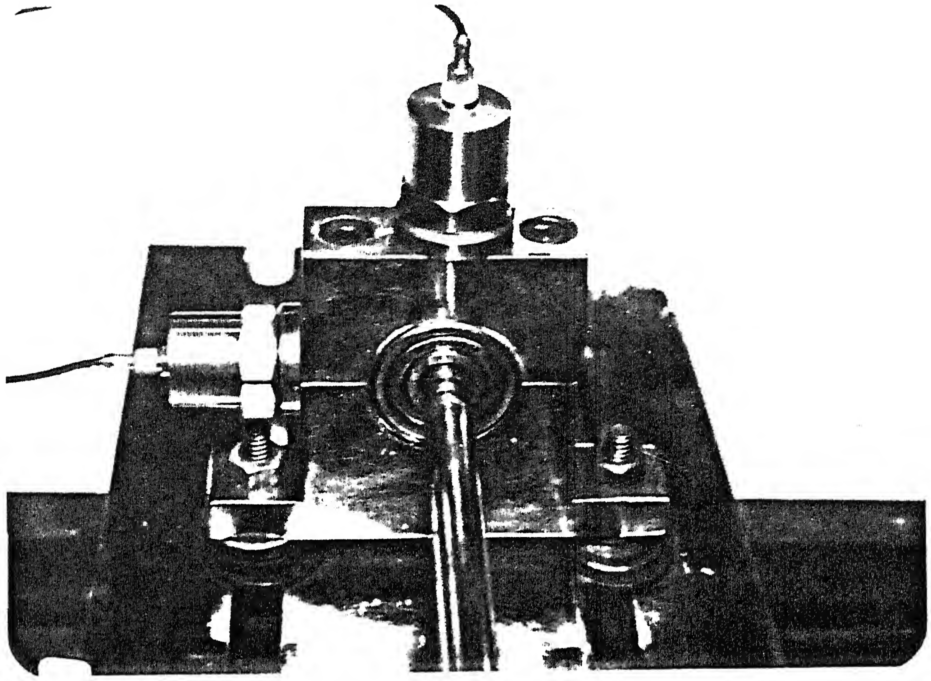


Figure 3.4 Accelerometer mountings on the bearing cap

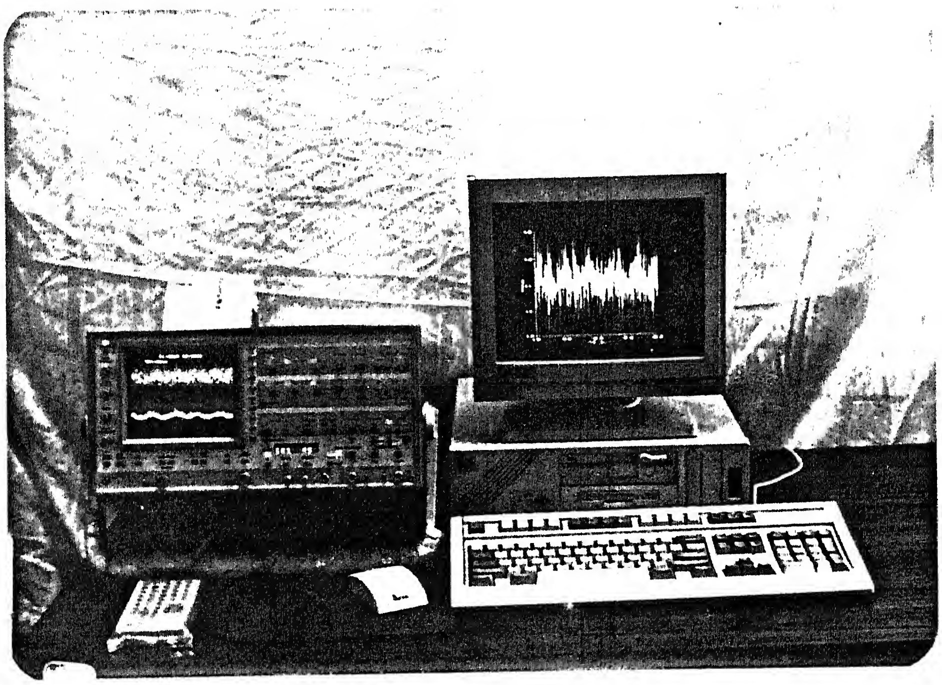


Figure 3.5 Digital storage oscilloscope connected to PC/AT with GPIB

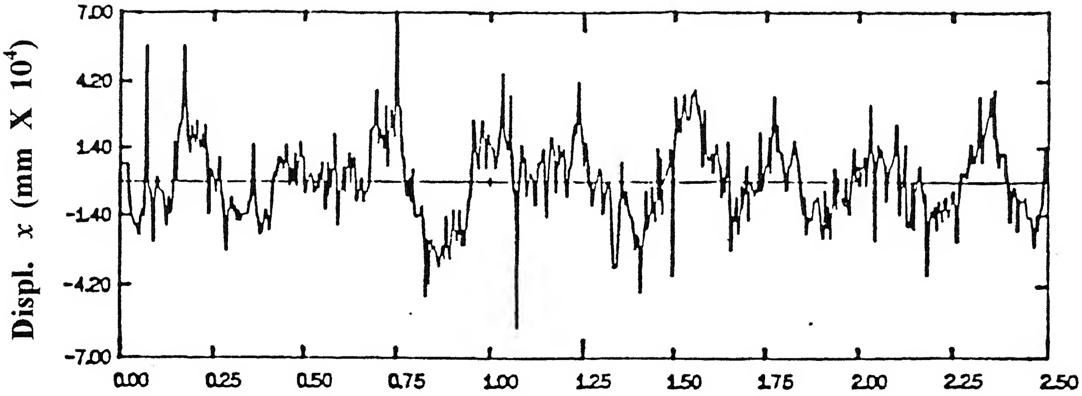


Figure 3.6 **Displacement signal in the vertical direction**

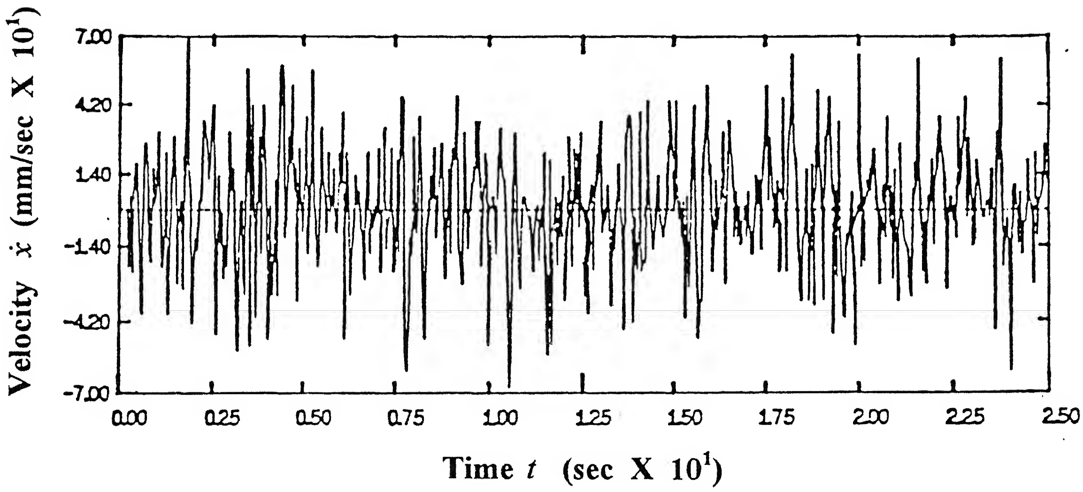


Figure 3.7 **Velocity signal in the vertical direction**

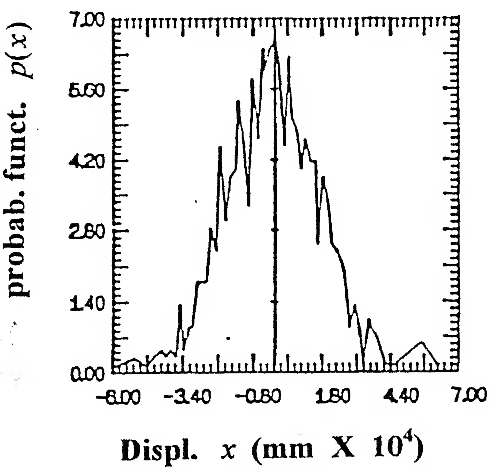


Figure 3.8 **Probability density distribution of vertical displacement**

3.6 MONTE CARLO SIMULATION

The algorithm is tested by Monte Carlo simulation, for the assumption involved. The values of ω_n^2 and λ, estimated from the experimental data, are fed into equation (3.1). A broad band excitation force, *f(t)*, with zero mean and Gaussian probability distribution, typically described in Figures 3.9 and 3.10 is computationally simulated. This force is also fed into equation (3.1) and the displacement and velocity responses, x and $\dot{x}$, are numerically obtained through fourth order Runga-Kutta numerical technique. This simulated vertical displacement and velocity response is shown in Figures 3.11 and 3.12. The probability distribution of the simulated vertical displacement is shown in Figure 3.13. Parameter estimation, as described in the above subsection is now carried out with the simulated response, by feeding it into equation (3.24) to obtain the values of ω_n^2 and λ. A similar exercise is carried out to obtain the parameters in the horizontal direction. These values are listed in Table 3.1.

Table 3.1 Estimated and Simulated Bearing Stiffness Parameters

	Parameters			
	Experimental		Simulated	
	ω_n^2 (rads/sec)^2	λ (mm^{-2})	ω_n^2 (rads/sec)^2	λ (mm^{-2})
Vertical	5.42 x 10^7	- 1.27 x 10^6	5.42 x 10^7	- 1.29 x 10^6
Horizontal	3.21 x 10^7	- 1.29 x 10^6	3.21 x 10^7	- 1.23 x 10^6

The good agreement between the values of the bearing stiffness parameters, ω_n^2 and λ, obtained by processing the experimental data and those from the Monte Carlo simulation, indicate the correctness of the experimental and algebraic exercises. It should be noted that the simulated values of the bearing stiffness parameters are obtained for an ideal white noise excitation, while the experimental ones are obtained by processing the actual response of the system, where the unknown excitation was idealized as white noise. It also needs to be pointed out that the value of the damping ratio, ς, is not required for the estimation procedure (equation (3.24)) . Any convenient value of ς can be employed in equation (3.1) for the purpose of simulation ($\varsigma = 0.02$ has been assumed in the present simulation).

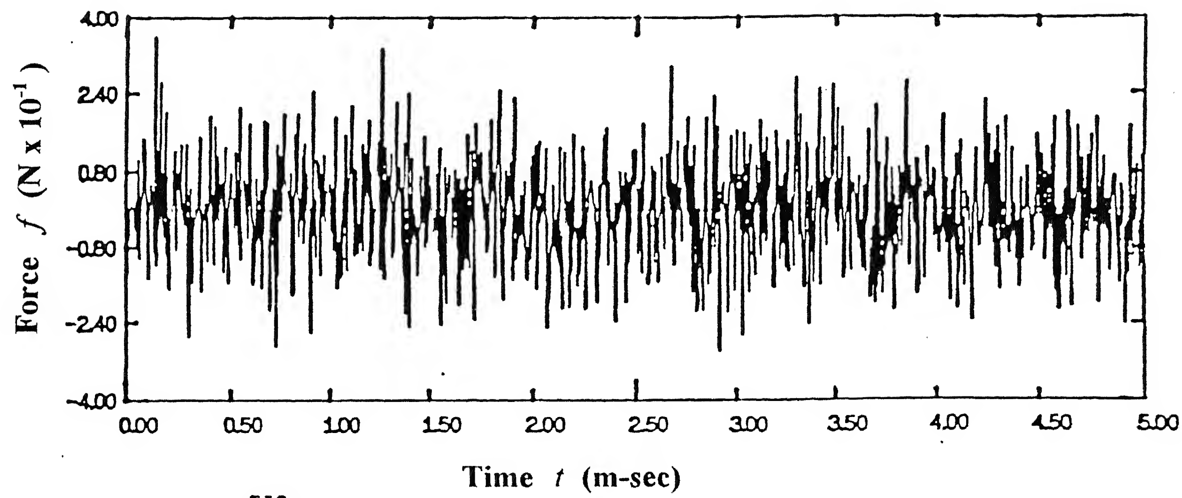


FIG. .

Figure 3.9 Simulated random force

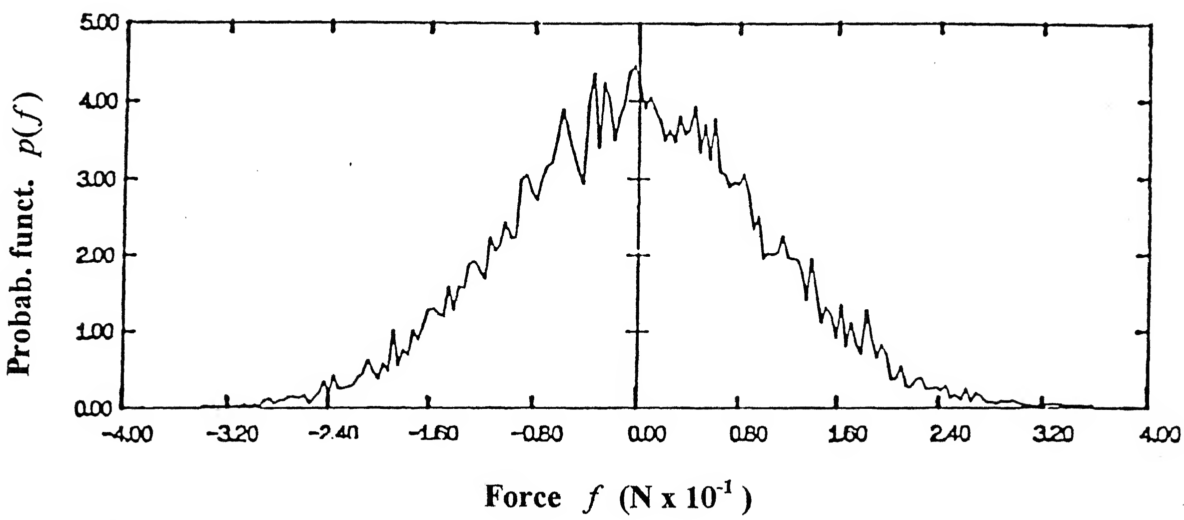
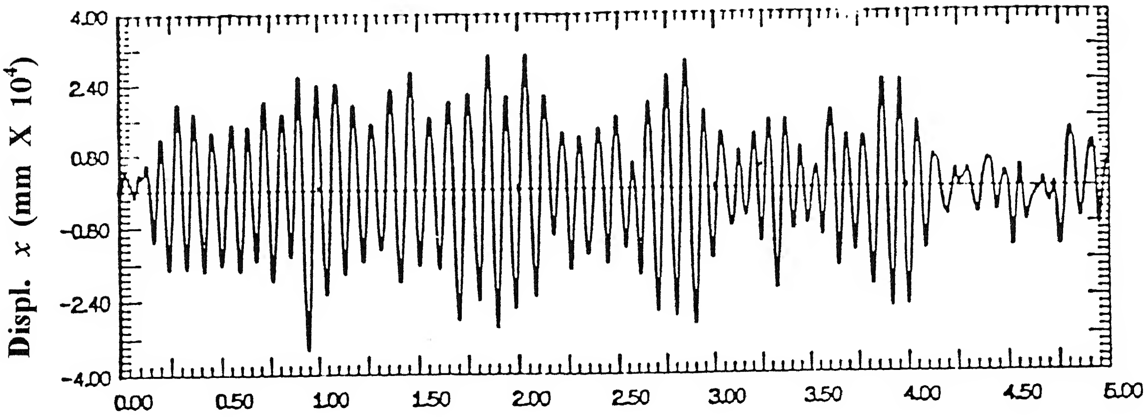


Figure 3.10 Probability density distribution of simulated random force

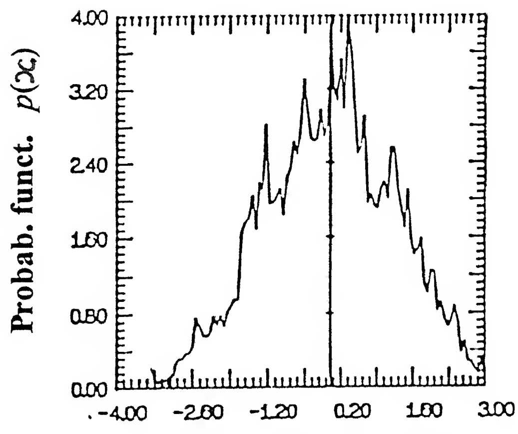


Time t (sec X 10^1)

Figure 3.11 **Simulated displacement signal**

Time t (sec X 10^1)

Figure 3.12 **Simulated velocity signal**



Displ. x (mm X 10^4)

Figure 3.13 **Probability density distribution of simulated vertical displacement**

3.7 VALIDATION

The values of the bearing stiffness parameters ω_n^2 and λ, obtained by the procedure outlined are compared with those obtained from the analytical formulations of Harris (1984) and Ragulskis et al. (1974), which are based on Hertzian contact theory.

For a given external radial load R on a bearing (Figure 3.14), the total elastic force at the points of contact of the ith ball with the inner and outer races is expressed as (Ragulskis et al., 1974)

$$f_i = K_n(g + x\cos\eta_i + y\sin\eta_i)^{3/2} \tag{3.25}$$

and its projection along the line of action of the applied force is

$$F_i = K_n(g + x\cos\eta_i + y\sin\eta_i)^{3/2}\cos\eta_i \tag{3.26}$$

where g is the radial preload or preclearance between the ball and the races and x and y are the displacements of the moving ring in the direction of the radial load and perpendicular to the direction of the radial load respectively. η_i is the angle between the lines of action of the radial load (direction of displacement of the moving ring) and the radius passing through the center of the ith ball. K_n is a coefficient of proportionality depending on the geometric and material properties of the bearing. The specifications of the test bearing are given below.

Ball bearing type	SKF 6200
Number of balls	6
Ball diameter	6 mm
Bore diameter	10 mm
Outer diameter	30 mm
Pitch diameter	20 mm
Inner ring ball race radius	3.09 mm
Outer ring ball race radius	3.09 mm
Allowable pre-load	0 - 2 μm
Rotor mass per bearing	0.41 Kgs

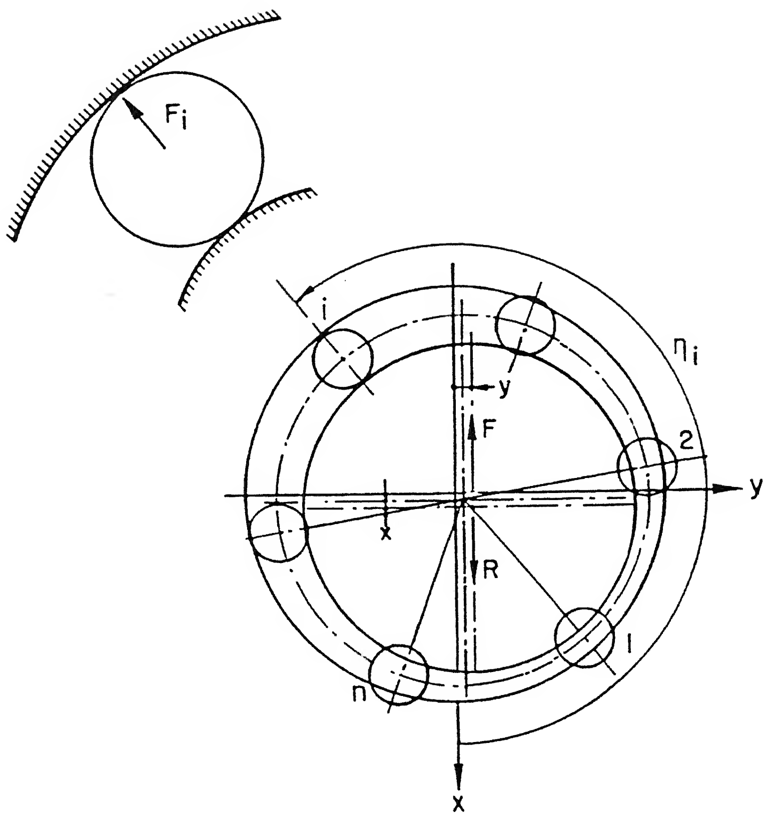


Figure 3.14 **Line diagram of loaded bearing**

The value of K_n, for the test bearing with above specifications, is estimated by the method suggested by Harris (1984) as 2.82×10^{5} N/mm$^{1.5}$.

The total elastic force in the direction of the applied force is

$$F = \sum_{i=1}^{n} F_i \tag{3.27}$$

where n is the total number of balls in the bearing.

Using the condition of zero elastic force in the direction perpendicular to the elastic load, the deformation, y, perpendicular to the radial force line is expressed as

$$y = \frac{\sum_{i=1}^{n}[g + x\cos(\eta_i)]^{3/2}\sin(\eta_i)}{\sum_{i=1}^{n}[g + x\cos(\eta_i)]^{1/2}\sin^2(\eta_i)} \tag{3.28}$$

Equations (3.26) and (3.28) are used in equation (3.27) and the bearing stiffness is determined as a function of the deformation x as

$$k(x) = \partial F / \partial x \tag{3.29}$$

Substituting equation (3.27) in equation (3.29), taking into account equation (3.28) the bearing stiffness is expressed as a function of deformation as

$$k(x) = K_n \sum_{i=1}^{n} [g + x\cos\eta_i - (A/Bn)\sin\eta_i]^{1/2} [\cos\eta_i - \{CBn - AD(n-1)\}/(Bn)^2 \sin\eta_i]\cos\eta_i$$

where

$$A = \sum_{i=1}^{n}[g + x\cos\eta_i]^{3/2}\sin\eta_i; \qquad B = \sum_{i=1}^{n}[g + x\cos\eta_i]^{1/2}\sin^2\eta_i$$

$$C = \sum_{i=1}^{n}[g + x\cos\eta_i]^{1/2}\sin\eta_i\cos\eta_i; \qquad D = \sum_{i=1}^{n}[g + x\cos\eta_i]^{-1/2}\sin^2\eta_i\cos\eta_i; \tag{3.30}$$

It can be seen that the bearing stiffness is critically dependent on the preloading, g, of the balls. While the manufacturer, may, at times, provide the preload range, the exact value of the preloading of the bearing balls in the shaft-casing assembly, especially during operations which have involved wear and tear, would be difficult to determine. The stiffness of the test bearing is plotted in Figure 3.15 as a function of the radial deformation, x, for various allowable preload values, g. The bearing stiffness obtained experimentally, using the procedure developed, also shown in Figure 3.15, shows good resemblance to theoretically possible values. It is also to be noted that the theoretical stiffness calculations are based on formulations which analyse the bearing in isolation of the shaft. The comparison between the experimental and theoretically possible stiffnesses is also listed in Table 3.2. The expressions for the theoretical stiffnesses in Table 3.2 have been obtained by curve fitting the stiffness values obtained from equation (3.30), through a quadratic in x.

Table 3.2 Estimated and Theoretical (Ragulskis et al., 1974; Harris, 1984) Bearing Stiffness Parameters

Preload (mm)	**Theoretical Stiffness (Radial) (N/mm)**	**Estimated Stiffness (N/mm)**
0.0002	$1.20 \times 10^4 - 4.01 \times 10^{10} x^2$	
		$1.32 \times 10^4 - 5.08 \times 10^{10} x^2$ **(horizontal)**
0.0003	$1.47 \times 10^4 - 2.18 \times 10^{10} x^2$	
0.0004	$1.69 \times 10^4 - 1.42 \times 10^{10} x^2$	
0.0005	$1.89 \times 10^4 - 1.02 \times 10^{10} x^2$	
		$2.23 \times 10^4 - 8.50 \times 10^{10} x^2$ **(vertical)**
0.0006	$2.08 \times 10^4 - 6.09 \times 10^9 x^2$	

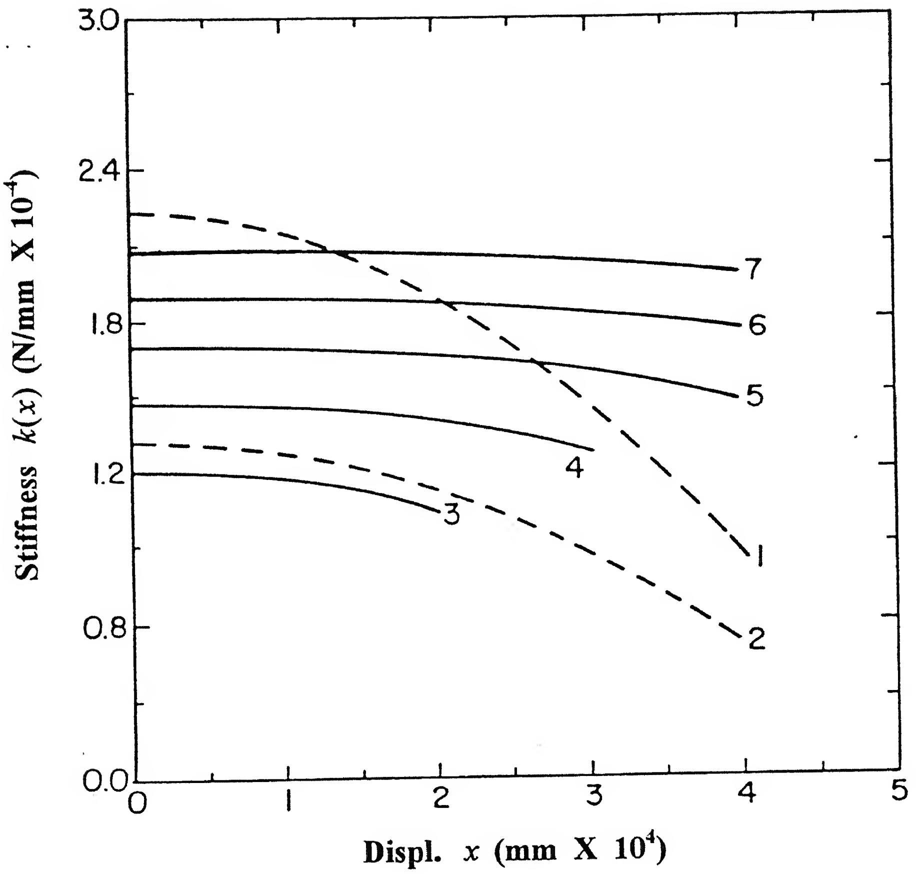


Figure 3.15 **Stiffness comparison**

1,2 - Present study (In vertical and horizontal directions respectively)
3,4,5,6,7 - Harris (1984) and Ragulskis et al. (1974) with prelaod 0.0002, 0.0003, 0.0004, 0.0005 and 0.0006 mm. respectively (In any radial direction)

3.8 REMARKS

The stiffness parameter estimates, from the procedure developed, show good agreement with the analytical formulations available for isolated ball bearings. The advantage of the proposed methods over other available techniques is distinct from the fact, that it does not involve measurement of the excitation forces and works directly on the random response signals, which can be conveniently picked up at the bearing caps and also that the procedure does not require the damping in the system to be known.

CHAPTER 4

BEARING STIFFNESS ESTIMATION IN FLEXIBLE SINGLE-DISC ROTORS

The bearing stiffness estimation procedure, described in Chapter 3, becomes more involved, if the shaft flexibility is taken into account. In contrast to the rigid rotor case, which could be treated as a single degree freedom problem, the flexible rotor poses a nonlinear multi-degree of freedom problem. The excitation to the balanced system is taken to be random in nature, primarily arising out of bearing and assembly imperfections. The problem is formulated for a flexible shaft carrying a centrally located rigid disc and supported at the ends by nonlinear bearings

4.1 EQUATIONS OF MOTION

A balanced rotor, with a centrally located disc on a massless flexible shaft supported in bearings at ends is shown in Figure 4.1. The shaft is treated as free-free body, carrying unknown effective bearing masses m_1 and m_2 at its ends and the known disc mass m_3 in the center. The bearings are incorporated through external "*forces*", F_b , acting on masses m_1 and m_2. Taking the shaft stiffness and damping forces as F_s and F_d , respectively, the equations of motion are written as

$$-F_{d_1} - F_{s_1} - F_{b_1} = m_1 \ddot{x}_1 \tag{4.1}$$

$$-F_{d_2} - F_{s_2} - F_{b_2} = m_2 \ddot{x}_2 \tag{4.2}$$

$$-F_{d_3} - F_{s_3} = m_3 \ddot{x}_3 \tag{4.3}$$

The shaft stiffness forces are

$$F_{s_1} = \left(k_{11}x_1 + k_{12}x_2 + k_{13}x_3\right)$$

$$F_{s_2} = \left(k_{12}x_1 + k_{22}x_2 + k_{23}x_3\right)$$

$$F_{s_3} = \left(k_{13}x_1 + k_{23}x_2 + k_{33}x_3\right) \tag{4.4}$$

The shaft stiffness parameter k_{ij} is defined as the *i*th force corresponding to a unit *j* deflection with all other deflections held to zero and can be obtained from Strength of Materials formulae.

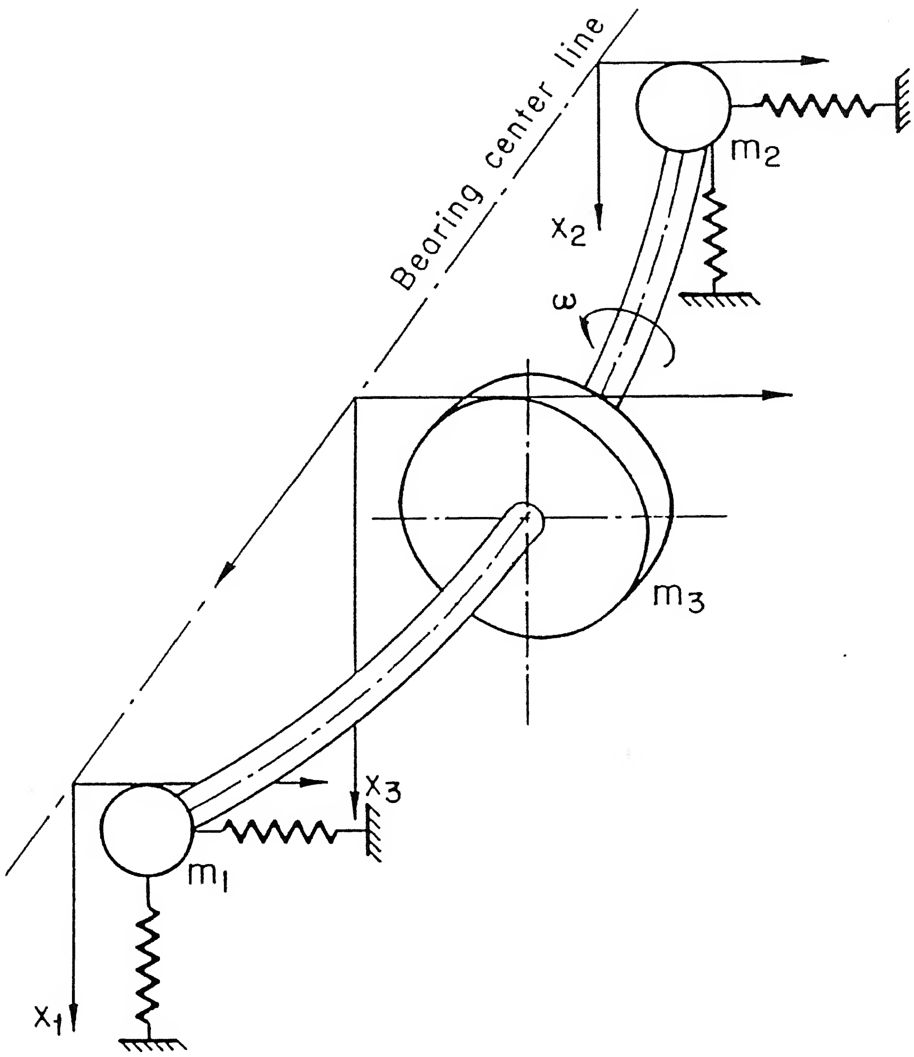


Figure 4.1 **Flexible rotor on rolling element bearings**

Bearing center line

The shaft damping forces are

$$F_{d_1} = (\alpha_{11}\dot{x}_1 + \alpha_{12}\dot{x}_2 + \alpha_{13}\dot{x}_3)$$
$$F_{d_2} = (\alpha_{12}\dot{x}_1 + \alpha_{22}\dot{x}_2 + \alpha_{23}\dot{x}_3)$$
$$F_{d_3} = (\alpha_{13}\dot{x}_1 + \alpha_{23}\dot{x}_2 + \alpha_{33}\dot{x}_3) \tag{4.5}$$

Shaft damping parameter, α_{ij} , can be defined in a manner similar to k_{ij} .

The excitation to the system is taken to be random in nature, as in the previous chapter, mainly arising out of the bearing surface imperfections, caused by the random deviations from their standard theoretical design and progressive surface and subsurface deteriorations. Sources such as inaccuracies in the rotor-bearing-housing assembly etc. can also contribute to the random excitation.

Taking s_1 and s_2 as the effective random displacements at the bearings, primarily due to surface imperfections and inaccuracies in the rotor-bearing-housing assembly, the bearing forces on masses m_1 and m_2 can be written as

$$F_{b_1} = \left\{k_{L_1}(x_1 - s_1) + k_{NL_1}G(x_1 - s_1)\right\}$$
$$F_{b_2} = \left\{k_{L_2}(x_2 - s_2) + k_{NL_2}G(x_2 - s_2)\right\} \tag{4.6}$$

In the above . k_L and k_{NL} are the unknown linear and nonlinear bearing stiffness parameters and G can be a polynomial in x .

Equations (4.6) can be rewritten, more generally, as

$$F_{b_1} = \left\{k_{L_1}x_1 + k_{NL_1}G(x_1)\right\} - F_1(t)$$
$$F_{b_2} = \left\{k_{L_2}x_2 + k_{NL_2}G(x_2)\right\} - F_2(t) \tag{4.7}$$

where $F_1(t)$ and $F_2(t)$ are the random components of the bearing forces.

Using (4.4), (4.5) and (4.7) the equations of motion (4.1-4.3), can be written as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} + \begin{Bmatrix} \partial V(x_1,x_2,x_3)/\partial x_1 \\ \partial V(x_1,x_2,x_3)/\partial x_2 \\ \partial V(x_1,x_2,x_3)/\partial x_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ 0 \end{Bmatrix} \tag{4.8}$$

where

$$V(x_1,x_2,x_3) = m_3\left[\frac{1}{2}\omega_{11}^2 x_1^2 + \frac{1}{2}\omega_{22}^2 x_2^2 + \frac{1}{2}\omega_{33}^2 x_3^2 + \omega_{12}^2 x_1 x_2 + \omega_{23}^2 x_2 x_3 + \omega_{13}^2 x_1 x_3\right]$$

$$+ m_3\left[\frac{1}{2}\omega_{n_1}^2 x_1^2 + \omega_{n_1}^2 \lambda_1 g(x_1) + \frac{1}{2}\omega_{n_2}^2 x_2^2 + \omega_{n_2}^2 \lambda_2 g(x_2)\right] \tag{4.9}$$

$$g(x) = \int_0^x G(\xi)d\xi \tag{4.10}$$

$$\omega_{ij}^2 = \frac{k_{ij}}{m_3}$$

$$\omega_{n_i}^2 = \frac{k_{L_i}}{m_3}$$

$$\lambda_i = \frac{k_{NL_i}}{\omega_{n_i}^2 m_3} \tag{4.11}$$

4.2 ORTHONORMAL TRANSFORMATION

The Markov vector approach extended to nonlinear multi-degree-of-freedom systems (Nigam, 1983) is adopted for the solution of equations (4.8).

The equations of motion, (4.8), with damping and forces F_1 and F_2 set to be zero, are solved for eigenvalues p_1^2, p_2^2, p_3^2 and orthonormal modal matrix $[U]$, so that

$$[U]^T\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix}[U]=\begin{bmatrix} p_1^2 & 0 & 0 \\ 0 & p_2^2 & 0 \\ 0 & 0 & p_3^2 \end{bmatrix} \tag{4.12}$$

$$[U]^T\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}[U]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.13}$$

(The eigenvalue and eigenvector matrix elements are given in Appendix -B.)

Application of coordinate transformation

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [U]\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} \tag{4.14}$$

to the equations of motion, (4.8), yields,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \\ \ddot{\eta}_3 \end{Bmatrix}+\begin{bmatrix} \beta_{11} & 0 & 0 \\ 0 & \beta_{22} & 0 \\ 0 & 0 & \beta_{33} \end{bmatrix}\begin{Bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{Bmatrix}+\begin{Bmatrix} \frac{1}{M_1}\partial V(\eta_1,\eta_2,\eta_3)/\partial\eta_1 \\ \frac{1}{M_2}\partial V(\eta_1,\eta_2,\eta_3)/\partial\eta_2 \\ \frac{1}{M_3}\partial V(\eta_1,\eta_2,\eta_3)/\partial\eta_3 \end{Bmatrix}=\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \tag{4.15}$$

where

$$\begin{bmatrix} \beta_{11} & 0 & 0 \\ 0 & \beta_{22} & 0 \\ 0 & 0 & \beta_{33} \end{bmatrix}=[U]^T\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix}[U] \tag{4.16}$$

$$V(\eta_1,\eta_2,\eta_3) = m_3\left[\frac{1}{2}\omega_{11}^2(\eta_1 + u_{31}\eta_2 + u_{21}\eta_3)^2 + \frac{1}{2}\omega_{22}^2(\eta_1 + u_{32}\eta_2 + u_{22}\eta_3)^2\right.$$

$$+\frac{1}{2}\omega_{33}^2(\eta_1 + \eta_2 + \eta_3)^2$$

$$+\omega_{12}^2(\eta_1 + u_{31}\eta_2 + u_{21}\eta_3)(\eta_1 + u_{32}\eta_2 + u_{22}\eta_3)$$

$$+\omega_{23}^2(\eta_1 + u_{32}\eta_2 + u_{22}\eta_3)(\eta_1 + \eta_2 + \eta_3)$$

$$\left.+\omega_{13}^2(\eta_1 + u_{31}\eta_2 + u_{21}\eta_3)(\eta_1 + \eta_2 + \eta_3)\right]$$

$$+m_3\left[\frac{1}{2}\omega_{n_1}^2(\eta_1 + u_{31}\eta_2 + u_{21}\eta_3)^2 + \omega_{n_1}^2\lambda_1 g(\eta_1 + u_{31}\eta_2 + u_{21}\eta_3)\right.$$

$$\left.+\frac{1}{2}\omega_{n_2}^2(\eta_1 + u_{32}\eta_2 + u_{22}\eta_3)^2 + \omega_{n_2}^2\lambda_2 g(\eta_1 + u_{32}\eta_2 + u_{22}\eta_3)\right] \quad (4.17)$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = [U]^T \begin{Bmatrix} F_1 \\ F_2 \\ 0 \end{Bmatrix} \quad (4.18)$$

and the modal masses M_1, M_2 and M_3 are given by

$$M_1 = m_1 + m_2 + m_3$$

$$M_2 = u_{31}^2 m_1 + m_2 + u_{32}^2 m_3$$

$$M_3 = u_{21}^2 m_1 + m_2 + u_{22}^2 m_3 \quad (4.19)$$

The approach to obtaining the response of the system is simplified, as in Chapter 3, by making an engineering assumption that random excitation to the system is such that the generalized forces q_i, in equations (4.15) can be treated as ideal white noise. An argument, similar to that in Chapter 3, can be given regarding this idealization, that, it turns out that the response obtained through such models are quite acceptable if the time scale of the excitation is much smaller than the time scale of the response (Lin, 1973). The excitation of equation (4.15), is treated as uncorrelated Gaussian, white random forces with the following properties

$$E[q_1(t)] = 0\,; \qquad E[q_2(t)] = 0\,; \qquad E[q_3(t)] = 0$$

and

$$E[q_1(t_1)q_1(t_2)] = 2\pi\phi_1\delta(t_2 - t_1)$$
$$E[q_2(t_1)q_2(t_2)] = 2\pi\phi_2\delta(t_2 - t_1) \qquad (4.20)$$
$$E[q_3(t_1)q_3(t_2)] = 2\pi\phi_3\delta(t_2 - t_1)$$

where ϕ_1, ϕ_2 and ϕ_3 denote the excitation intensity factors.

4.3 F-P-K EQUATION

Rewriting equations (4.15) as

$$\ddot{\eta}_i + \beta_{ii}\eta + (1/M_i)\partial V(\eta_1, \eta_2, \eta_3)/\partial\eta_i = 0 \qquad i = 1, 2, 3 \qquad (4.21)$$

and then, in state space form as

$$\begin{aligned} \partial\eta_i / \partial t &= \dot{\eta}_i \\ \partial\dot{\eta}_i / \partial t &= q_i - \beta_{ii}\eta_i - (1/M_i)(\partial V/\partial\eta_i) \end{aligned} \qquad (4.22)$$

the drift and diffusion coefficients (refer equations (3.5 - 3.7)), for equations (4.22) can be written as

$$\begin{aligned} a_1 &= \dot{\eta}_1 \qquad a_2 = \dot{\eta}_2 \qquad a_3 = \dot{\eta}_3 \\ b_1 &= q_1 - \beta_{11}\dot{\eta}_1 - (1/M_1)[\partial V(\eta_1, \eta_2, \eta_3)/\partial\eta_1] \\ b_2 &= q_2 - \beta_{22}\dot{\eta}_2 - (1/M_2)[\partial V(\eta_1, \eta_2, \eta_3)/\partial\eta_2] \\ b_3 &= q_3 - \beta_{33}\dot{\eta}_3 - (1/M_3)[\partial V(\eta_1, \eta_2, \eta_3)/\partial\eta_3] \end{aligned} \qquad (4.23)$$

and

$$\begin{aligned} &c_{11} = 0 \qquad c_{22} = 0 \qquad c_{33} = 0 \\ &\mathrm{d}_{11} = 2\pi\phi_1 \qquad \mathrm{d}_{22} = 2\pi\phi_2 \qquad \mathrm{d}_{33} = 2\pi\phi_3 \\ &e_{11} = e_{22} = e_{33} = e_{12} = e_{23} = e_{31} = 0 \end{aligned} \qquad (4.24)$$

With the help of the above, the joint probability density function, $p(\eta_1,\eta_2,\eta_3,\dot\eta_1,\dot\eta_2,\dot\eta_3)$, for the motion governed by equations (4.15) can be described by the Fokker-Planck equation

$$-\dot\eta_1\frac{\partial p}{\partial\eta_1}-\frac{1}{M_1}\frac{\partial V}{\partial\eta_1}\frac{\partial p}{\partial\dot\eta_1}+\frac{\partial}{\partial\dot\eta_1}\left(\beta_{11}\dot\eta_1 p+\pi\phi_1\frac{\partial p}{\partial\dot\eta_1}\right)-\dot\eta_2\frac{\partial p}{\partial\eta_2}-\frac{1}{M_2}\frac{\partial V}{\partial\eta_2}\frac{\partial p}{\partial\dot\eta_2}$$
$$+\frac{\partial}{\partial\dot\eta_2}\left(\beta_{22}\dot\eta_2 p+\pi\phi_2\frac{\partial p}{\partial\dot\eta_2}\right)-\dot\eta_3\frac{\partial p}{\partial\eta_3}-\frac{1}{M_3}\frac{\partial V}{\partial\eta_3}\frac{\partial p}{\partial\dot\eta_3}+\frac{\partial}{\partial\dot\eta_3}\left(\beta_{33}\dot\eta_3 p+\pi\phi_3\frac{\partial p}{\partial\dot\eta_3}\right)=\frac{\partial p}{\partial t} \tag{4.25}$$

4.4 RESPONSE

For a stationary case equation (4.25) reduces to

$$-\dot\eta_1\frac{\partial p}{\partial\eta_1}-\frac{1}{M_1}\frac{\partial V}{\partial\eta_1}\frac{\partial p}{\partial\dot\eta_1}+\frac{\partial}{\partial\dot\eta_1}\left(\beta_{11}\dot\eta_1 p+\pi\phi_1\frac{\partial p}{\partial\dot\eta_1}\right)-\dot\eta_2\frac{\partial p}{\partial\eta_2}-\frac{1}{M_2}\frac{\partial V}{\partial\eta_2}\frac{\partial p}{\partial\dot\eta_2}$$
$$+\frac{\partial}{\partial\dot\eta_2}\left(\beta_{22}\dot\eta_2 p+\pi\phi_2\frac{\partial p}{\partial\dot\eta_2}\right)-\dot\eta_3\frac{\partial p}{\partial\eta_3}-\frac{1}{M_3}\frac{\partial V}{\partial\eta_3}\frac{\partial p}{\partial\dot\eta_3}+\frac{\partial}{\partial\dot\eta_3}\left(\beta_{33}\dot\eta_3 p+\pi\phi_3\frac{\partial p}{\partial\dot\eta_3}\right)=0 \tag{4.26}$$

With an assumption (Caughey, 1963)
$(\beta_{11}/M_1\phi_1)=(\beta_{22}/M_2\phi_2)=(\beta_{33}/M_3\phi_3)=\gamma$

equation (4.26) can be solved to obtain the joint probability density of displacements and velocities in terms of the transformed coordinate system as

$$p(\eta_1,\eta_2,\eta_3,\dot\eta_1,\dot\eta_2,\dot\eta_3)=c\exp\left[-\frac{\gamma}{\pi}\left\{M_1\dot\eta_1^2/2+M_2\dot\eta_2^2/2+M_3\dot\eta_3^2/2+V(\eta_1,\eta_2,\eta_3)\right\}\right] \tag{4.27}$$

Performing the inverse orthonormal transformation and noting that the term

$$M_1\dot\eta_1^2/2+M_2\dot\eta_2^2/2+M_3\dot\eta_3^2/2+V(\eta_1,\eta_2,\eta_3)$$

on the right hand side of equation (4.27) represents the total kinetic and potential energy of the system and that γ is a constant, the joint probability density of displacements and velocities in the original set of coordinates is

$$p(x_1,x_2,x_3,\dot{x}_1,\dot{x}_2,\dot{x}_3) = c\exp\left[-\frac{\gamma}{\pi}\left\{m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2 + V(x_1,x_2,x_3)\right\}\right] \tag{4.28}$$

The joint probability functions $p(x_1,x_2,x_3)$ and $p(\dot{x}_1,\dot{x}_2,\dot{x}_3)$ are obtained from equation (4.28) as (Roberts and Spanos, 1990)

$$p(x_1,x_2,x_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x_1,x_2,x_3,\dot{x}_1,\dot{x}_2,\dot{x}_3)d\dot{x}_1 d\dot{x}_2 d\dot{x}_3$$

$$= c_1 \exp\left[-\frac{\gamma}{\pi}V(x_1,x_2,x_3)\right]$$

with

$$c_1 = 1\Big/\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{\gamma}{\pi}V(x_1,x_2,x_3)\right]dx_1 dx_2\, dx_3 \tag{4.29}$$

and

$$p(\dot{x}_1,\dot{x}_2,\dot{x}_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x_1,x_2,x_3,\dot{x}_1,\dot{x}_2,\dot{x}_3)dx_1 dx_2 dx_3$$

$$= \left[\frac{1}{\pi}\sqrt{\frac{\gamma}{2}}\right]^3 \sqrt{m_1 m_2 m_3}\exp\left[-\frac{\gamma}{\pi}\left\{\frac{1}{2}\left(m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2\right)\right\}\right] \tag{4.30}$$

The joint probability function $p(x_1,x_2)$ is obtained as

$$p(x_1,x_2) = \int_{-\infty}^{\infty} p(x_1,x_2,x_3)dx_3$$

$$= \int_{-\infty}^{\infty} c_1 \exp\left[-\frac{\gamma}{\pi}\{V(x_1,x_2,x_3\}\right]dx_3$$

$$= c_2 \exp\left[-\frac{m_3\gamma}{\pi} \left\{ \frac{1}{2}\left(\omega_{11}^2 + \omega_{n_1}^2 - \frac{\omega_{13}^4}{\omega_{33}^2} \right) x_1^2 + \frac{1}{2}\left(\omega_{22}^2 + \omega_{n_2}^2 - \frac{\omega_{23}^4}{\omega_{33}^2} \right) x_2^2 \right.\right.$$

$$\left.\left. + \left(\omega_{12}^2 - \frac{\omega_{13}^2 \omega_{23}^2}{\omega_{33}^2} \right) x_1 x_2 + (\omega_{n_1}^2 \lambda_1) g(x_1) + (\omega_{n_2}^2 \lambda_2) g(x_2) \right\} \right]$$

with

$$c_2 = \left(\frac{\pi}{\omega_{33}} \sqrt{\frac{2}{m_3\gamma}} \right) \Bigg/ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\frac{\gamma}{\pi} V(x_1, x_2, x_3) \right] dx_1 dx_2\, dx_3 \tag{4.31}$$

The probability density function $p(\dot{x}_1)$ and $p(\dot{x}_2)$ are

$$p(\dot{x}_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\dot{x}_1, \dot{x}_2, \dot{x}_3) d\dot{x}_2 d\dot{x}_3$$

$$= \left[\frac{1}{\pi} \sqrt{\frac{m_1\gamma}{2}} \right] \exp\left[-\frac{\gamma}{\pi} \left\{ \frac{1}{2} m_1 \dot{x}_1^2 \right\} \right] \tag{4.32}$$

$$p(\dot{x}_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\dot{x}_1, \dot{x}_2, \dot{x}_3) d\dot{x}_1 d\dot{x}_3$$

$$= \left[\frac{1}{\pi} \sqrt{\frac{m_2\gamma}{2}} \right] \exp\left[-\frac{\gamma}{\pi} \left\{ \frac{1}{2} m_2 \dot{x}_2^2 \right\} \right] \tag{4.33}$$

The variances of velocity responses $\dot{x}_1$ and $\dot{x}_2$ are obtained as

$$\sigma_{\dot{x}_1}^2 = \int_{-\infty}^{\infty} \dot{x}_1^2 p(\dot{x}_1) d\dot{x}_1$$

$$= \frac{\pi}{m_1\gamma} \tag{4.34}$$

$$\sigma_{\dot{x}_2}^2 = \int_{-\infty}^{\infty} \dot{x}_2^2 p(\dot{x}_2) d\dot{x}_2$$

$$= \frac{\pi}{m_2\gamma} \tag{4.35}$$

Combining equations (4.34) and (4.35)

$$\sigma_{\dot{x}_1}\sigma_{\dot{x}_2} = \frac{\pi}{\gamma}\frac{1}{\sqrt{m_1 m_2}} \tag{4.36}$$

The joint probability density function for the displacement responses x_1 and x_2, from equations (4.31) and (4.36), can be written as

$$p(x_1,x_2) = c_2 \exp\left[-\frac{\sqrt{\mu_1\mu_2}}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}}\left\{\frac{1}{2}\left(\omega_{11}^2+\omega_{n_1}^2-\frac{\omega_{13}^4}{\omega_{33}^2}\right)x_1^2+\frac{1}{2}\left(\omega_{22}^2+\omega_{n_2}^2-\frac{\omega_{23}^4}{\omega_{33}^2}\right)x_2^2\right.\right.$$
$$\left.\left.+\left(\omega_{12}^2-\frac{\omega_{13}^2\omega_{23}^2}{\omega_{33}^2}\right)x_1x_2+(\omega_{n_1}^2\lambda_1)g(x_1)+(\omega_{n_2}^2\lambda_2)g(x_2)\right\}\right]$$

with

$$c_2 = \left(\frac{\pi}{\omega_{33}}\sqrt{\frac{2}{m_3\gamma}}\right)\Bigg/\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{\sqrt{\mu_1\mu_2}}{m_3\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}}V(x_1,x_2,x_3)\right]dx_1dx_2\,dx_3$$

$$\mu_1 = \frac{m_3}{m_1}\ ; \qquad \mu_2 = \frac{m_3}{m_2} \tag{4.37}$$

4.5 EXTRACTION OF BEARING STIFFNESS PARAMETERS

The bearing parameters are obtained from the experimentally obtained random response in terms of the linear stiffness parameters $\omega_{n_1}^2, \omega_{n_2}^2$ and the nonlinear stiffness parameters λ_1, λ_2 and the disc-bearing mass ratios μ_1 and μ_2. These parameters are obtained for both, the vertical and horizontal directions, the problem formulation, in the horizontal direction, remaining identical to that in the vertical direction.

The joint probability density function $p(x_1,x_2)$ for a set of displacements (x_{1_i},x_{2_j}) $(x_{1_{(i+1)}},x_{2(j+1)})$, $(x_{1_{(i+1)}}\rangle x_{1_i}$ and $x_{2_{(j+1)}}\rangle x_{2_j})$, from equation (4.37) are

$$p(x_{1_i},x_{2_j}) = c_2 \exp\left[-\frac{\sqrt{\mu_1\mu_2}}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}}\left\{\frac{1}{2}\left(\omega_{11}^2+\omega_{n_1}^2-\frac{\omega_{13}^4}{\omega_{33}^2}\right)x_{1_i}^2+\frac{1}{2}\left(\omega_{22}^2+\omega_{n_2}^2-\frac{\omega_{23}^4}{\omega_{33}^2}\right)x_{2_j}^2\right.\right.$$
$$\left.\left.+\left(\omega_{12}^2-\frac{\omega_{13}^2\omega_{23}^2}{\omega_{33}^2}\right)x_{1_i}x_{2_j}+(\omega_{n_1}^2\lambda_1)g(x_{1_i})+(\omega_{n_2}^2\lambda_2)g(x_{2_j})\right\}\right] \tag{4.38}$$

$$p(x_{1_{(i+1)}},x_{2_{(j+1)}}) = c_2 \exp\left[-\frac{\sqrt{\mu_1\mu_2}}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}}\left\{\frac{1}{2}\left(\omega_{11}^2+\omega_{n_1}^2-\frac{\omega_{13}^4}{\omega_{33}^2}\right)x_{1_{(i+1)}}^2+\frac{1}{2}\left(\omega_{22}^2+\omega_{n_2}^2-\frac{\omega_{23}^4}{\omega_{33}^2}\right)x_{2_{(j+1)}}^2\right.\right.$$
$$\left.\left.+\left(\omega_{12}^2-\frac{\omega_{13}^2\omega_{23}^2}{\omega_{33}^2}\right)x_{1_{(i+1)}}x_{2_{(j+1)}}+(\omega_{n_1}^2\lambda_1)g(x_{1_{(i+1)}})+(\omega_{n_2}^2\lambda_2)g(x_{2_{(j+1)}})\right\}\right]$$

Defining

$$\Delta x_{1_i} = x_{1_{(i+1)}} - x_{1_i}\ ; \qquad \Delta x_{2_j} = x_{2_{(j+1)}} - x_{2_j}$$

for small Δx_{1_i} and Δx_{2_j}, one can write

$$p(x_{1_{(i+1)}},x_{2_{(j+1)}}) = c_2 \exp\left[-\frac{\sqrt{\mu_1\mu_2}}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}}\left\{\frac{1}{2}\left(\omega_{11}^2+\omega_{n_1}^2-\frac{\omega_{13}^4}{\omega_{33}^2}\right)x_{1_i}^2+\frac{1}{2}\left(\omega_{22}^2+\omega_{n_2}^2-\frac{\omega_{23}^4}{\omega_{33}^2}\right)x_{2_j}^2\right.\right.$$
$$\left.\left.+\left(\omega_{12}^2-\frac{\omega_{13}^2\omega_{23}^2}{\omega_{33}^2}\right)x_{1_i}x_{2_j}+(\omega_{n_1}^2\lambda_1)g(x_{1_i})+(\omega_{n_2}^2\lambda_2)g(x_{2_j})\right\}\right]\cdot$$
$$\exp\left[-\frac{\sqrt{\mu_1\mu_2}}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}}\left\{\left(\omega_{11}^2+\omega_{n_1}^2-\frac{\omega_{13}^4}{\omega_{33}^2}\right)(x_{1_i}\Delta x_{1_i})+\left(\omega_{22}^2+\omega_{n_2}^2-\frac{\omega_{23}^4}{\omega_{33}^2}\right)(x_{2_j}\Delta x_{2_j})\right.\right.$$
$$\left.\left.+\left(\omega_{12}^2-\frac{\omega_{13}^2\omega_{23}^2}{\omega_{33}^2}\right)(x_{1_i}\Delta x_{2_j}+x_{2_j}\Delta x_{1_i})+(\omega_{n_1}^2\lambda_1)G(x_{1_i})\Delta x_{1_i}+(\omega_{n_2}^2\lambda_2)G(x_{2_j})\Delta x_{2_j}\right\}\right] \tag{4.39}$$

Combining equations (4.39) and (4.38) gives

$$p(x_{1_{(i+1)}}, x_{2_{(j+1)}}) =$$

$$p(x_{1_i}, x_{2_j}) \exp\left[-\frac{\sqrt{\mu_1\mu_2}}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}} \left\{ \left(\omega_{11}^2 + \omega_{n_1}^2 - \frac{\omega_{13}^4}{\omega_{33}^2} \right)(x_{1_i}\Delta x_{1_i}) + \left(\omega_{22}^2 + \omega_{n_2}^2 - \frac{\omega_{23}^4}{\omega_{33}^2} \right)(x_{2_j}\Delta x_{2_j}) \right.\right.$$

$$\left.\left. + \left(\omega_{12}^2 - \frac{\omega_{13}^2\omega_{23}^2}{\omega_{33}^2} \right)(x_{1_i}\Delta x_{2_j} + x_{2_j}\Delta x_{1i}) + (\omega_{n_1}^2\lambda_1)G(x_{1_i})\Delta x_{1_i} + (\omega_{n_2}^2\lambda_2)G(x_{2_j})\Delta x_{2_j} \right\} \right]$$

(4.40)

For N values of each displacements, $x_{1_1}, x_{1_2}, \ldots\ldots\ldots, x_{1_N}$, and $x_{2_1}, x_{2_2}, \ldots\ldots\ldots, x_{2_N}$, equation (4.40) can be expressed as a set of $(N-1)^2$ linear simultaneous algebraic equations, as,

$$\left[\frac{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}}{\Delta x_{1_i}\Delta x_{2_j}} \ln\left\{ \frac{p(x_{1_i}, x_{2_j})}{p(x_{1_{(i+1)}}, x_{2_{(j+1)}})} \right\} \right] \left\{ \frac{1}{\sqrt{\mu_1\mu_2}\,\omega_{n_1}^2} \right\} - \left[\frac{G(x_{1_i})}{\Delta x_{2_j}} \right] \{\lambda_1\} - \left[\frac{G(x_{2_j})}{\Delta x_{1_i}} \right] \left\{ \frac{\lambda_2\omega_{n_2}^2}{\omega_{n_1}^2} \right\}$$

$$+ \left[\left(\frac{\omega_{13}^4}{\omega_{33}^2} - \omega_{11}^2 \right)\left(\frac{x_{1_i}}{\Delta x_{2_j}} \right) + \left(\frac{\omega_{23}^4}{\omega_{33}^2} - \omega_{22}^2 \right)\left(\frac{x_{2_j}}{\Delta x_{1_i}} \right) + \left(\frac{\omega_{13}^2\omega_{23}^2}{\omega_{33}^2} - \omega_{12}^2 \right)\left(\frac{x_{1_i}}{\Delta x_{1_i}} + \frac{x_{2_j}}{\Delta x_{2_j}} \right) \right] \left\{ \frac{1}{\omega_{n_1}^2} \right\}$$

$$+ \left[\left(\frac{\omega_{23}^4}{\omega_{33}^2} - \omega_{11}^2 \right)\left(\frac{x_{2_j}}{\Delta x_{1_i}} \right) \right] \left\{ \frac{\left(1 - \dfrac{\omega_{n_2}^2}{\omega_{n_1}^2} \right)}{\left(\dfrac{\omega_{23}^4}{\omega_{33}^2} - \omega_{22}^2 \right)} \right\} = \left(\frac{x_{1_i}}{\Delta x_{2_j}} + \frac{x_{2_j}}{\Delta x_{1_i}} \right) \qquad (4.41)$$

$i = 1,2,\ldots\ldots\ldots,(N-1)$; $\qquad j = 1,2,\ldots\ldots\ldots,(N-1)$

Equations (4.41) are solved for $\omega_{n_1}^2$, $\omega_{n_2}^2, \lambda_1, \lambda_2$, and $\sqrt{\mu_1\mu_2}$, using the least square fit technique. The joint probability function, $p(x_1, x_2)$ and variances, $\sigma_{\dot{x}_1}^2$ and $\sigma_{\dot{x}_2}^2$ are computed from the experimentally obtained displacement and velocity data ($x_1, x_2, \dot{x}_1$ and $\dot{x}_2$), which are taken as zero mean Gaussian processes and the nonlinear spring force provided by the rolling element bearings is taken to be cubic in nature i.e. $G(x) = x^3$. The stiffness-matrix defined as for the three body lumped-parameter shaft model (Figure 4.1) can be defined as (Childs, 1993)

$$[K] = m_3 \begin{bmatrix} \omega_{11}^2 & \omega_{13}^2 & \omega_{12}^2 \\ \omega_{31}^2 & \omega_{33}^2 & \omega_{32}^2 \\ \omega_{21}^2 & \omega_{23}^2 & \omega_{22}^2 \end{bmatrix} = \frac{12EI}{L^3} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad (4.42)$$

4.6 EXPERIMENTATION

The laboratory rig, described in Chapter 3, is employed for experimental illustration of the procedure. Typical displacement and velocity signals, in the vertical direction, picked up (after balancing the rotor) by the accelerometers mounted on the two bearing housings are given in Figures 4.2, 4.3, 4.4, and 4.5. The joint probability density function, $p(x_1,x_2)$, of the displacements is shown in Figure 4.6. The following set of data is taken for the rotor

$EI = 1.03 \times 10^8$ N-mm^2

$m_3 = 0.515$ kg

$L = 250.0$ mm.,

The parameters estimated from equations (4.41) are given in Table 4.1.

Table 4.1 Estimated bearing stiffness and mass parameters

Parameters	Vertical	Horizontal
$\omega_{n_1}^2$ **(rads/sec)^2**	2.13×10^7	2.12×10^7
λ_1 **(mm)$^{-2}$**	-1.25×10^6	-1.41×10^6
$\omega_{n_2}^2$ **(rads/sec)^2**	1.98×10^7	2.21×10^7
λ_2 **(mm)$^{-2}$**	-1.15×10^6	-1.36×10^6
$\sqrt{\mu_1\mu_2}$	2.10	2.34

(Refer to Appendix C for Statistical Error Table)

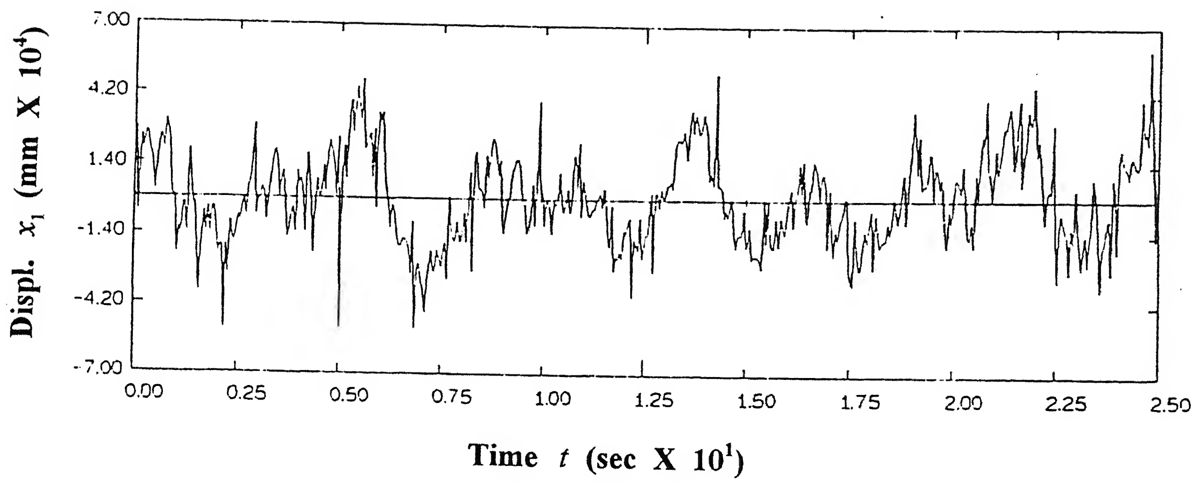


Figure 4.2 **Displacement signal in the vertical direction at bearing 1**

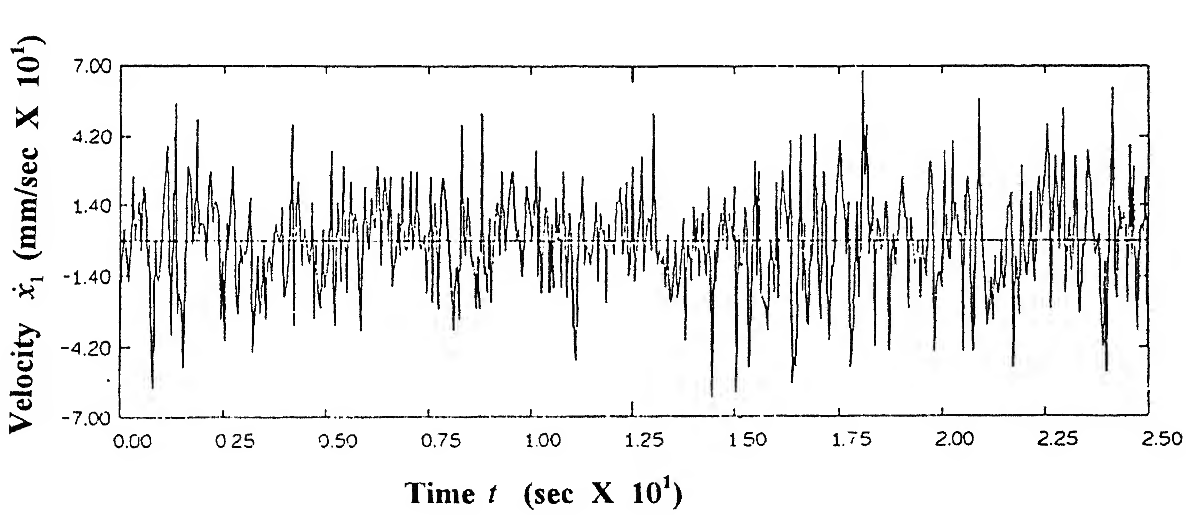


Figure 4.3 **Velocity signal in the vertical direction at bearing 1**

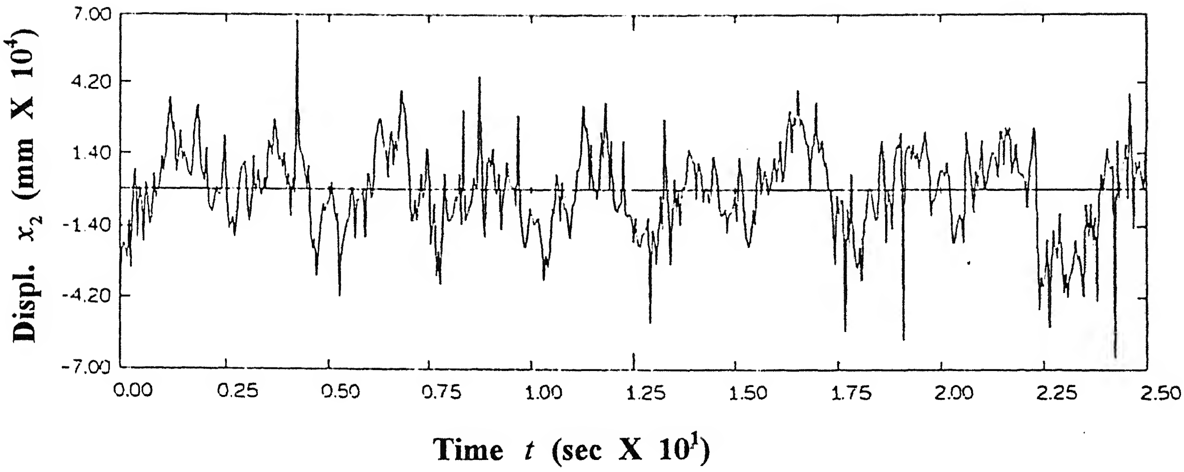


Figure 4.4 **Displacement signal in the vertical direction at bearing 2**

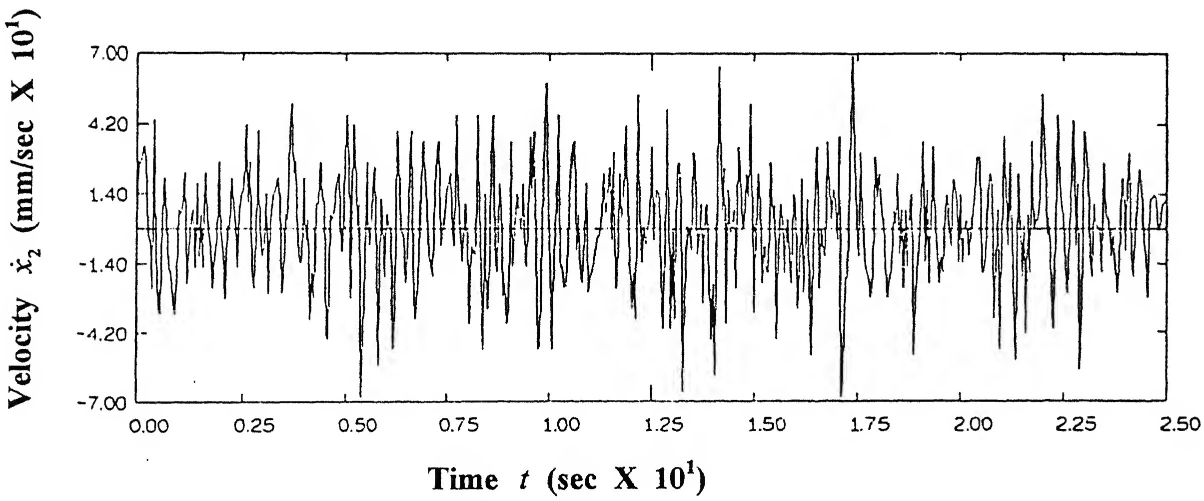


Figure 4.5 **Velocity signal in the vertical direction at bearing 2**

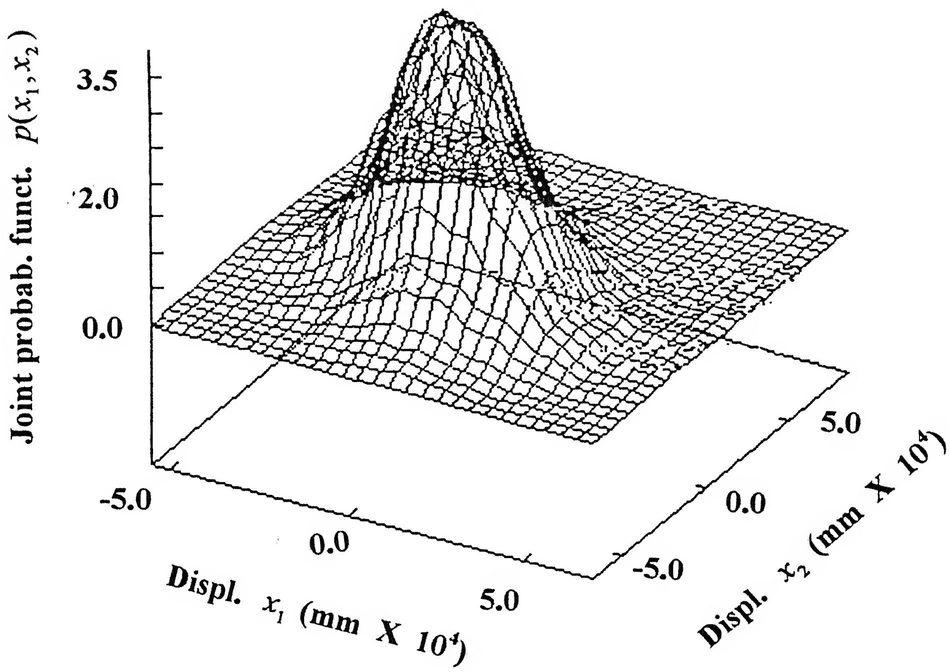


Figure 4.6 Joint probability density distribution of vertical displacements at bearing 1 and bearing 2

4.7 MONTE CARLO SIMULATION

The algorithm is tested by Monte Carlo simulation. The experimentally obtained values of $\omega^2_{n_1}, \omega^2_{n_2}, \lambda_1, \lambda_2$ and $\sqrt{\mu_1 \mu_2}$ are fed into equation (4.8). Broad band excitation forces, $f_1(t)$ and $f_2(t)$ with zero mean and Gaussian probability distribution as described in Figures 4.7, 4.8, 4.9 and 4.10, are computationally simulated. These forces are also fed into equation (4.8) and the equations are numerically solved through fourth order Runga-Kutta to obtain the displacement and velocity responses, $x_1, x_2, \dot{x}_1$ and $\dot{x}_2$. These simulated displacement and velocity responses, at the two bearings, are shown for the vertical direction in Figures 4.11, 4.12, 4.13 and 4.14. The joint probability distribution of the simulated vertical displacements is shown in Figure 4.15. The simulated response is now fed into equation (4.41) to obtain the values of $\omega^2_{n_1}, \omega^2_{n_2}, \lambda_1, \lambda_2$ and $\sqrt{\mu_1 \mu_2}$. A similar exercise is carried out to obtain the parameters in the horizontal direction. These values are listed in Table 4.2.

Table 4.2 Estimated and simulated bearing stiffness and mass parameters

	Vertical		Horizontal	
Parameters	**Experimental**	**Simulated**	**Experimental**	**Simulated**
$\omega^2_{n_1}$ **(rads/sec)^2**	2.13 x 10^7	2.12 x 10^7	2.11 x 10^7	2.21 x 10^7
λ_1 **(mm)$^{-2}$**	-1.25 x 10^6	-1.41 x 10^6	-1.29 x 10^6	-1.45 x 10^6
$\omega^2_{n_2}$ **(rads/sec)^2**	1.98 x 10^7	2.21 x 10^7	1.95 x 10^7	2.23 x 10^7
λ_2 **(mm)$^{-2}$**	-1.15 x 10^6	-1.36 x 10^6	-1.63 x 10^6	-1.86 x 10^6
$\sqrt{\mu_1 \mu_2}$	2.10	2.43	2.34	2.50

The good agreement between the values of the bearing stiffness parameters, $\omega^2_{n_1}, \omega^2_{n_2}, \lambda_1, \lambda_2$ and $\sqrt{\mu_1 \mu_2}$, obtained by processing the experimental data and those from the Monte Carlo simulation, indicate the correctness of the experimental and algebraic exercises. It should be noted that the simulated values of the bearing stiffness parameters are obtained for an ideal white noise

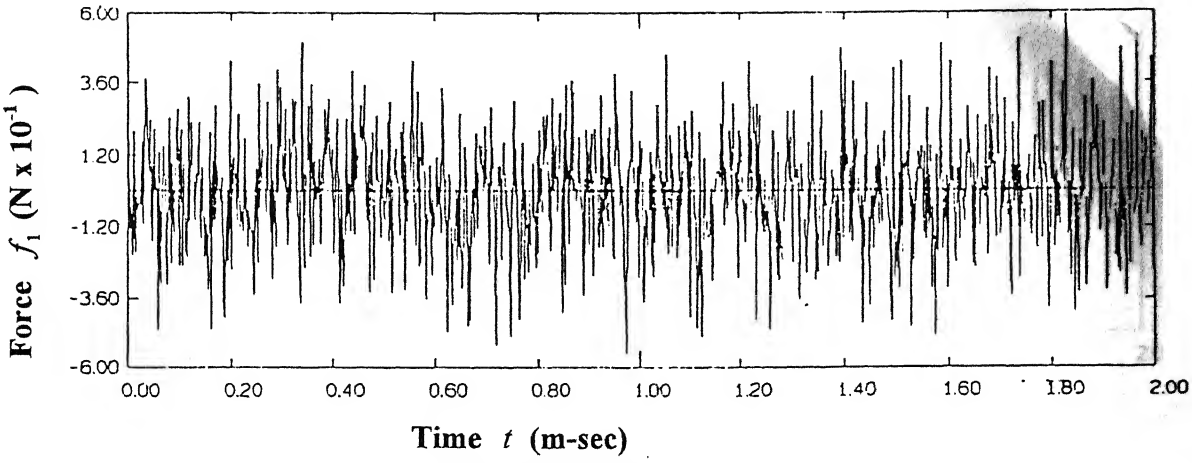


Figure 4.7 **Simulated random force at bearing 1**

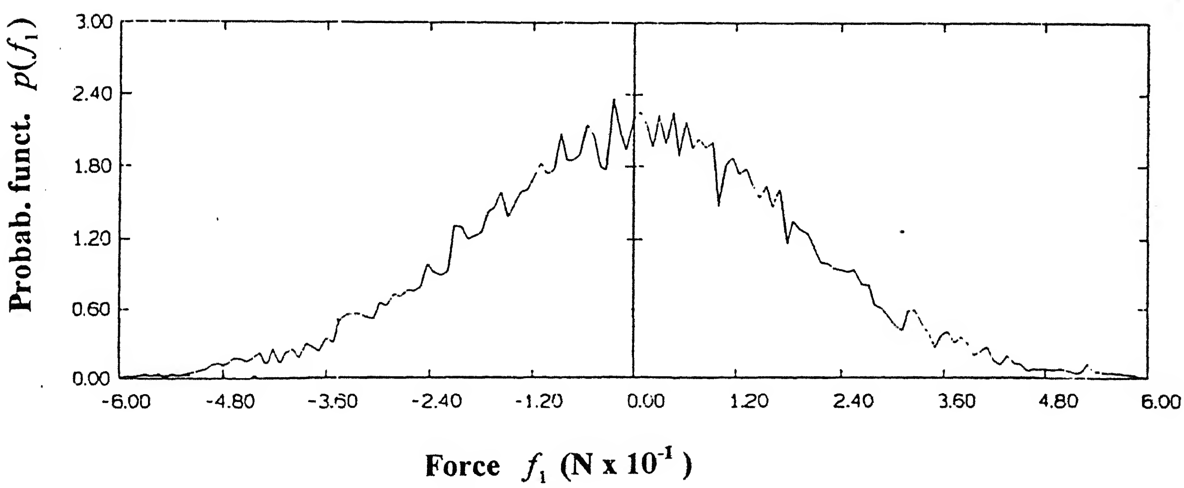


Figure 4.8 **Probability density distribution of simulated force at bearing 1**

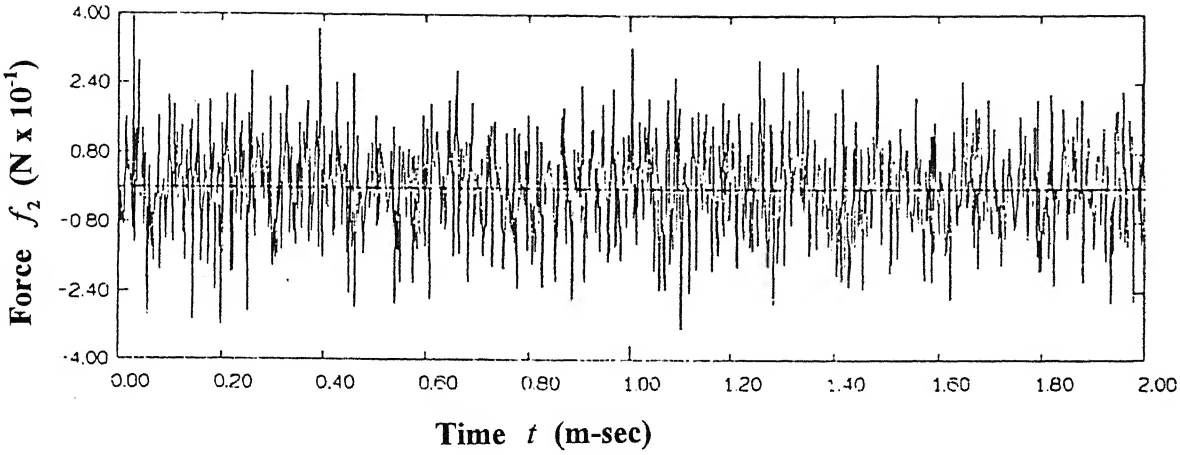


Figure 4.9 **Simulated random force at bearing 2**

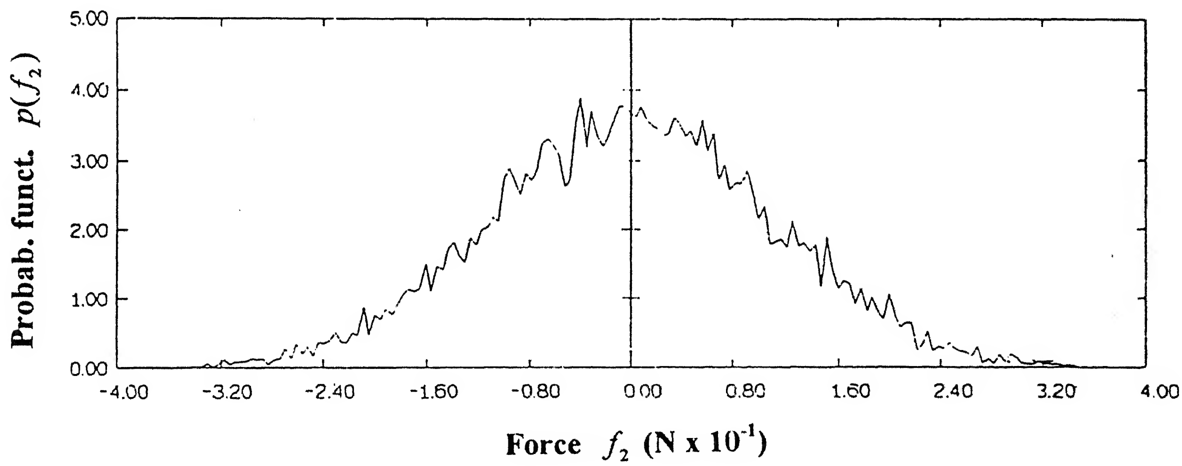


Figure 4.10 **Probability density distribution of simulated force at bearing 2**

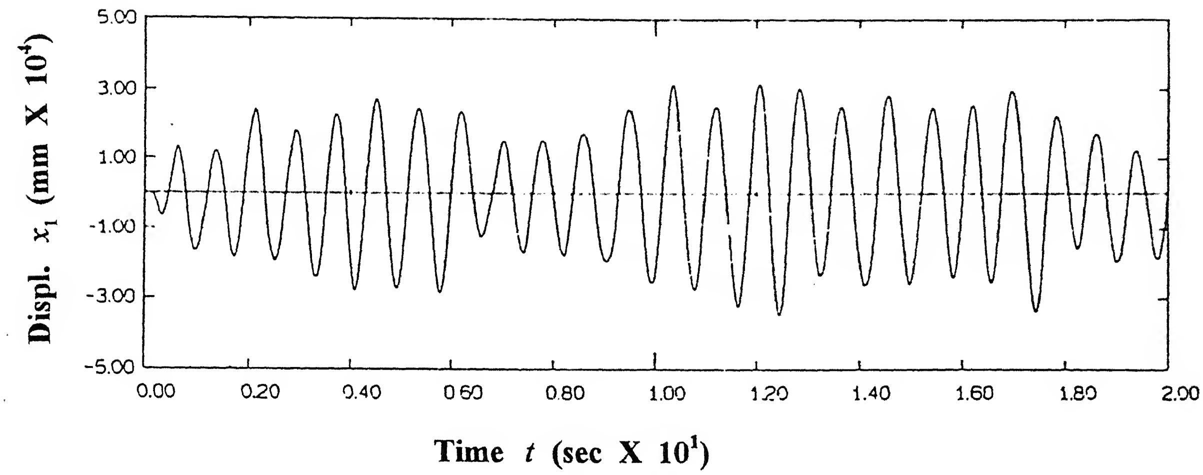


Figure 4.11 **Simulated displacement signal at bearing 1**

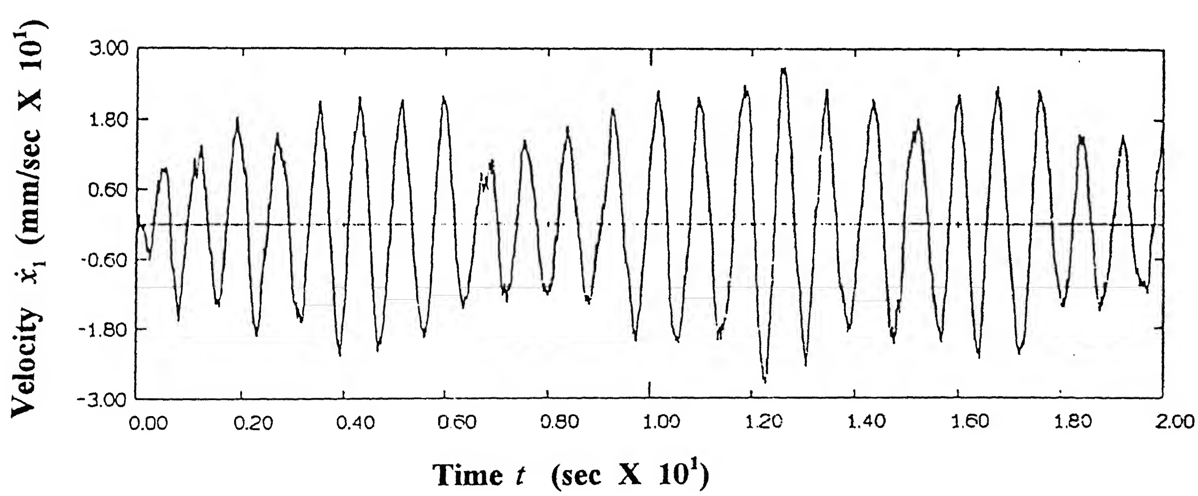


Figure 4.12 **Simulated velocity signal at bearing 1**

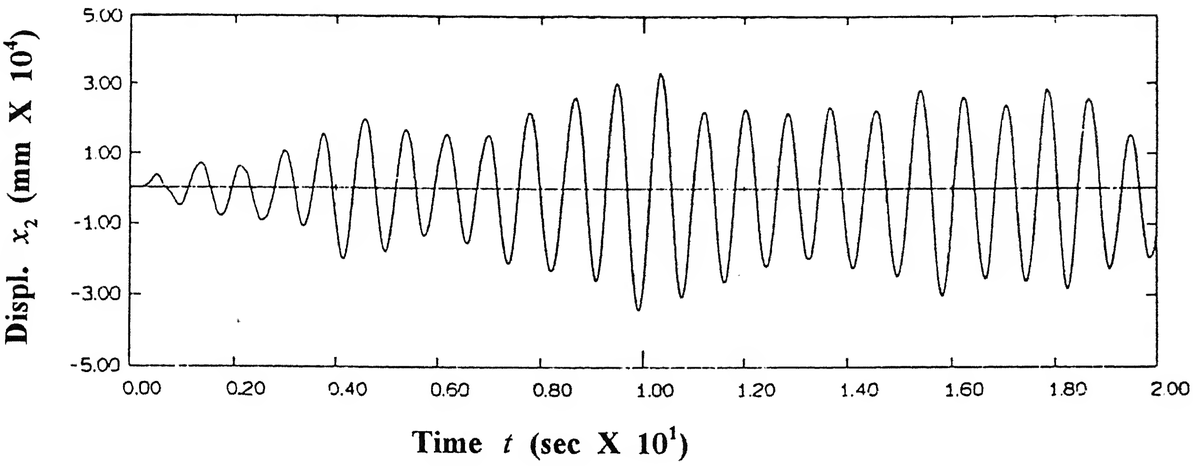


Figure 4.13 **Simulated displacement signal at bearing 2**

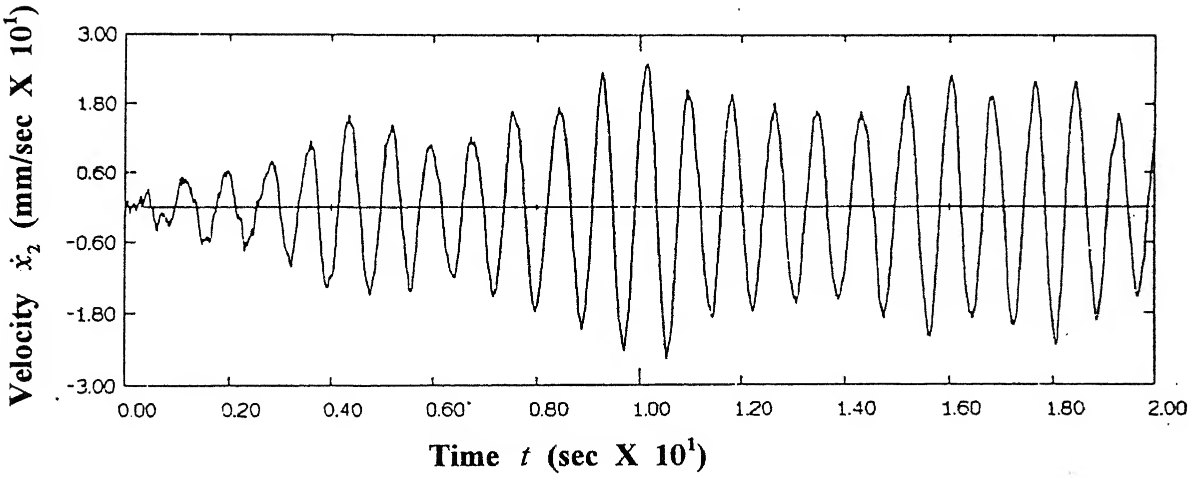


Figure 4.14 **Simulated velocity signal at bearing 2**

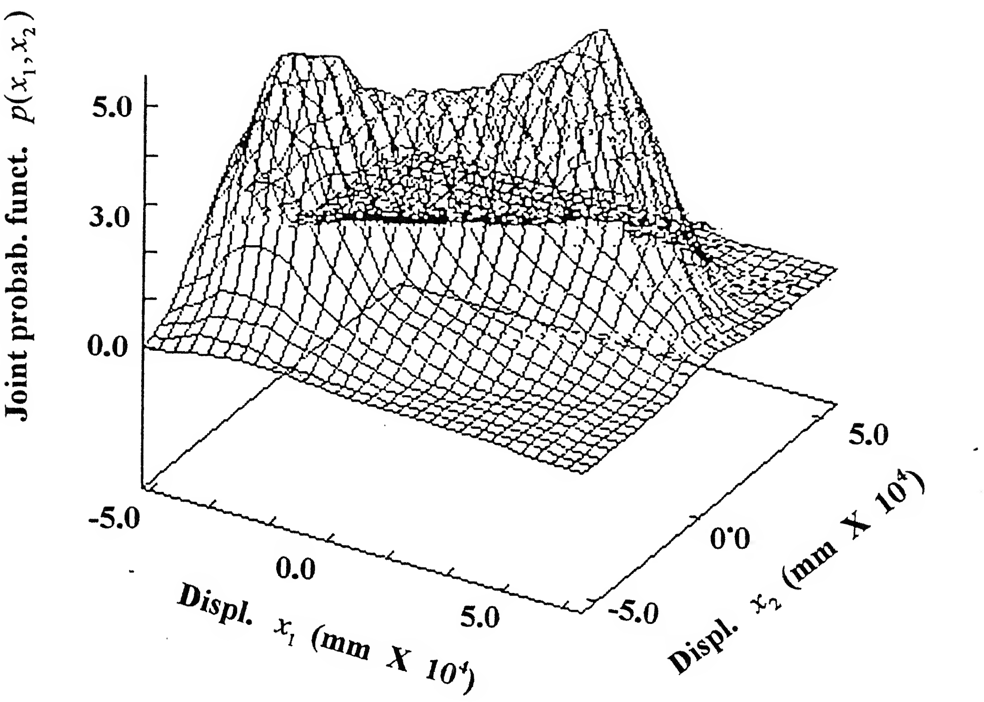


Figure 4.15 Joint probability density distribution of simulated vertical displacements at bearing 1 and bearing 2

excitation, while the experimental ones are obtained by processing the actual response of the system, where the unknown excitation was idealized as white noise. It also needs to be pointed out that the values of the damping parameters α_{ij}, are not required for the estimation procedure (equation (4.41)) . Any convenient set of values of α_{ij}, can be employed in equations (4.8-4.11) for the purpose of simulation.

4.8 VALIDATION

The analytical formulations of Ragulskis, et al. (1974) and Harris (1984), described earlier in Chapter 3, are used to validate the stiffness parameters obtained from the experimental signals, through the procedure developed. The bearing stiffness expression obtained from the analytical formulations given in the literature mentioned above, are listed in Table 4.3, under the column 'Theoretical Stiffness'. The stiffnesses estimated for the two bearings, through the procedure developed, are also listed in Table 4.3.

Table 4.3 Estimated and theoretical (Ragulskis, et al., 1974; Harris, 1984) bearing stiffness parameters

Preload (mm)	Theoretical Stiffness (Radial) (N/mm)	Estimated Stiffness (N/mm) (at bearing 1)	Estimated Stiffness (N/mm) (at bearing 2)
0.0002	$1.20 \times 10^4 - 4.01 \times 10^{10} x^2$		
		$1.08 \times 10^4 - 4.19 \times 10^{10} x^2$ **(horizontal)**	$1.01 \times 10^4 - 4.91 \times 10^{10} x^2$ **(horizontal)**
0.0003	$1.47 \times 10^4 - 2.18 \times 10^{10} x^2$		
0.0004	$1.69 \times 10^4 - 1.42 \times 10^{10} x^2$		
0.0005	$1.89 \times 10^4 - 1.02 \times 10^{10} x^2$		
		$1.10 \times 10^4 - 4.13 \times 10^{10} x^2$ **(vertical)**	$1.02 \times 10^4 - 3.52 \times 10^{10} x^2$ **(vertical)**
0.0006	$2.08 \times 10^4 - 6.09 \times 10^9 x^2$		

As emphasized in the earlier chapter, the bearing stiffness is critically dependent on the preloading, g, of the balls. The exact value of the preloading of the bearing balls in the shaft-casing assembly, especially during operations which have involved wear and tear, would be difficult to determine. The stiffness of the test bearing is plotted in Figures 4.16 and 4.17 as a function of the radial deformation, x, for various allowable preload values, g. The bearing stiffness obtained experimentally, using the procedure developed, also shown in Figures 4.16 and 4.17, shows good resemblance to theoretically possible values. It is also to be noted that the theoretical stiffness calculations are based on formulations which analyse the bearing in isolation of the shaft. The comparison between the experimental and theoretically possible stiffness is also listed in Table 4.3. The expressions for the theoretical stiffness in Table 4.3 have been obtained by curve fitting the stiffness values obtained from equation (4.41), through a quadratic in x.

In the analysis, presented in Chapter 3, where the shaft flexibility has not been accounted for and the shaft is treated as a rigid body, the bearing stiffness for the same experimental set-up was found to be $1.32 \times 10^4 - 5.08 \times 10^{10}\ x^2$ (N/mm) and $2.23 \times 10^4 - 8.50 \times 10^{10}\ x^2$ (N/mm) in the horizontal and vertical directions respectively (Refer Table 3.2). A comparison with the stiffness values of Table 4.3 reveals the influence of shaft flexibility.

While a good agreement on the bearing stiffness parameters is observed between the values generated following the method of Harris (1984) and Ragulskis et al. (1974) and those obtained experimentally through the present procedure, the values of the effective masses at the bearing ends, obtained as by-products of the present procedure, also look reasonable. The experimentally obtained values of $\sqrt{\mu_1 \mu_2}$ are 2.1 and 2.34 (Table 4.1) in the vertical and horizontal directions respectively. If the two bearings were taken to be identical, the effective mass computed (knowing that the disc mass is 0.515 Kgs) for each of the bearing ends turns out to be 0.245 Kgs in the vertical direction and 0.220 Kgs in the horizontal direction. These values look reasonable, in view of the fact that along with some contribution from the bearings themselves, a division of the mass of the shaft, which in this case is 0.306 Kgs, is seen at the two bearing ends.

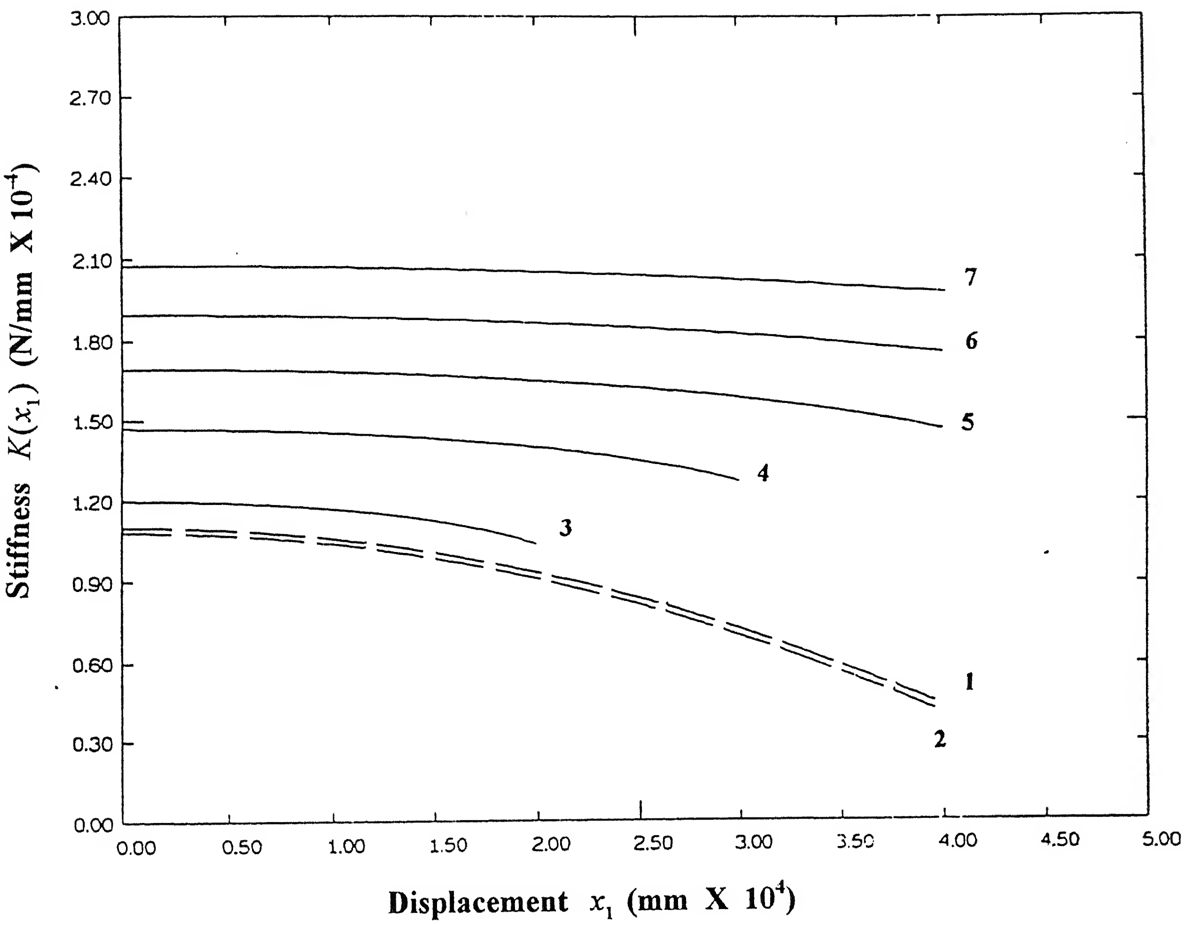


Figure 4.16 **Stiffness comparison at bearing 1**
1,2 - Present study (In vertical and horizontal directions respectively)
3,4,5,6,7 - Harris (1984) and Ragulskis et al. (1974) with prelaod 0.0002, 0.0003, 0.0004, 0.0005 and 0.0006 mm. respectively (In any radial direction)

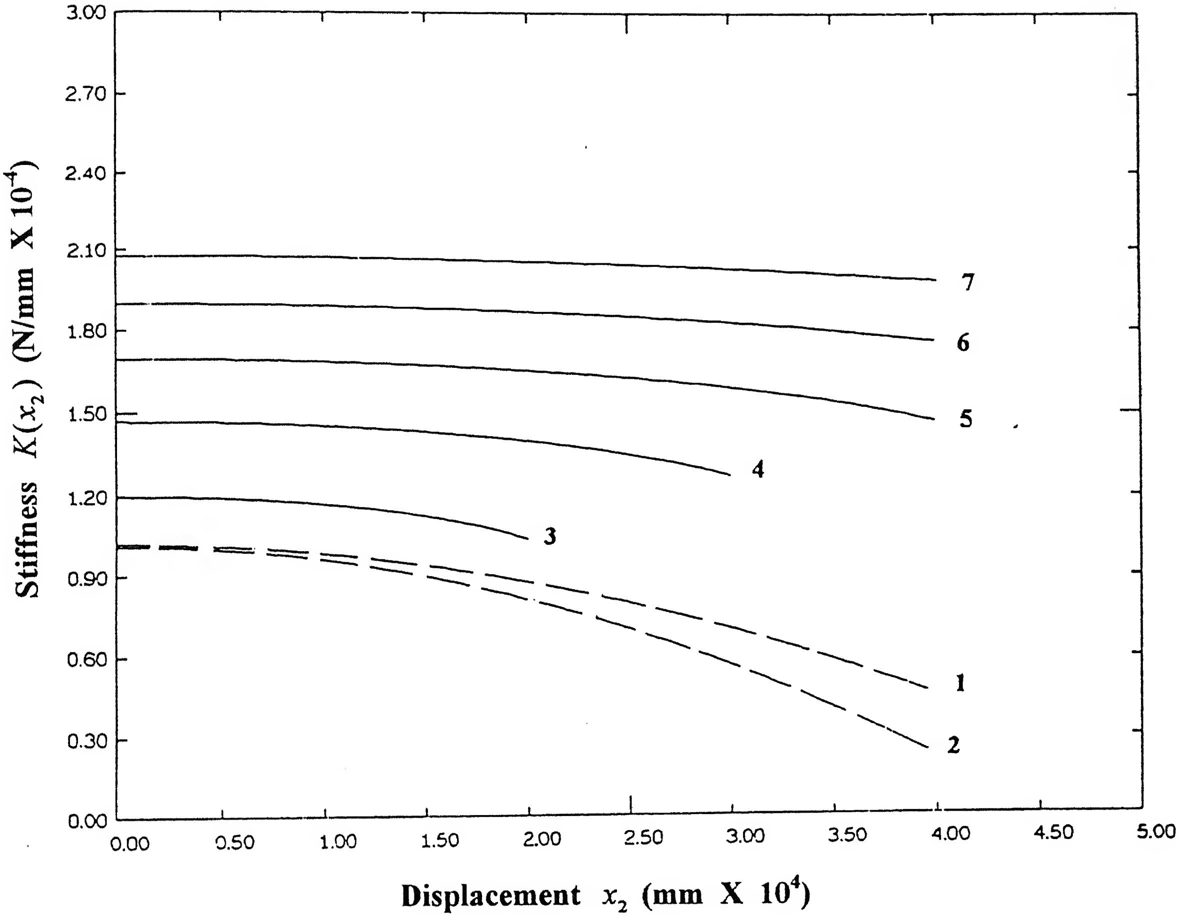


Figure 4.17 **Stiffness comparison at bearing 2**

1,2 - Present study (In vertical and horizontal directions respectively)
3,4,5,6,7 - Harris (1984) and Ragulskis et al. (1974) with prelaod 0.0002, 0.0003, 0.0004, 0.0005 and 0.0006 mm. respectively (In any radial direction)

4.9 REMARKS

The inverse problem for the flexible rotor, features a nonlinear multi-degree-freedom system and the essential feature, of the approach to the inverse problem of parameter estimation is an appropriate coordinate transformation, so as to enable the governing equations to be modeled as Markov Processes through the Fokker-Planck equations. In addition to the salient features of the estimation procedure already mentioned in the earlier chapter, accounting for the shaft flexibility, also enables to obtain estimates of the effective masses at the bearing stations.

CHAPTER 5

BEARING STIFFNESS ESTIMATION IN MULTI-DISC ROTORS

The parameter estimation problem in rotors carrying more than one disc on flexible shafts is considered next. The governing nonlinear differential equations for such multi-mass flexible systems are derived for balanced rotors with random excitation at the bearings. The governing equations are subjected to a coordinate transformation and modeled as Markov Processes. General form expressions, for the first order probability statistics of the response are obtained. The statistical response is processed to extract the rotor-bearing stiffness parameters.

5.1 EQUATIONS OF MOTION

A balanced rotor, with n discs mounted on a massless flexible shaft supported in nonlinear bearings at ends is shown in Figure 5.1. The shaft is treated as free-free body, carrying unknown effective bearing masses m_1 and m_2 at its ends and the known disc masses $m_3, m_4, \ldots, m_{n+2}$. Following the procedure mentioned in Chapter 4, for incorporating the bearings through external "*forces*", F_b , acting on masses m_1 and m_2 and taking the shaft parameters to be linear, the equations of motion are written as

$$[M]\{\ddot{X}\} + [A]\{\dot{X}\} + \{\partial V / \partial X\} = \{F\} \tag{5.1}$$

where $\{X\}$ is the displacement vector, $[M]$ and $[A]$ the mass and damping matrices respectively,

$$\{F\} = \{F_1 \;\; F_2 \;\; \ldots\ldots 0\}^T \tag{5.2}$$

$$V(x_1, x_2, \cdots, x_{(n+2)}) = (1/2)\{X\}^T([K]+[K_L])\{X\} + \{g(x)\}^T[K_{NL}]\{g(x)\} \tag{5.3}$$

$$\{g(x)\} = \{ \;\; (\int_0^{x_1} G(\xi)d\xi)^{1/2} \quad (\int_0^{x_2} G(\xi)d\xi)^{1/2} \quad \ldots\ldots\ldots\ldots \; 0 \;\}^T \tag{5.4}$$

$[K]$ is the shaft stiffness matrix and the linear and non-linear stiffness matrices of the bearing, respectively, are

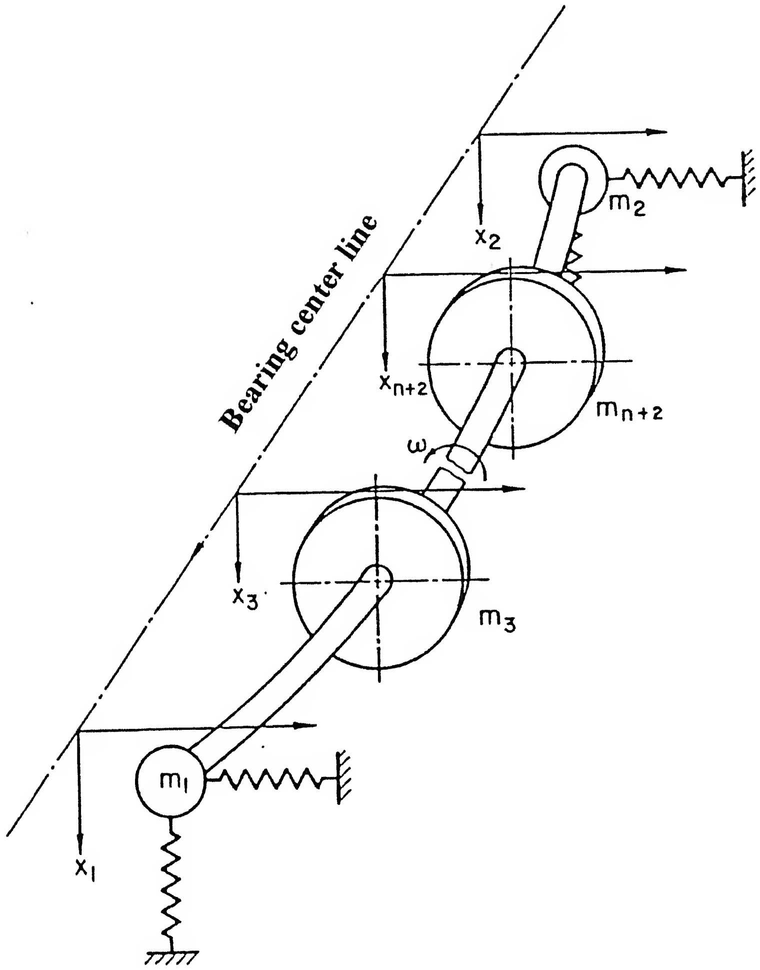


Figure 5.1 **Multi disc rotor on rolling element bearings**

$$[K_L] = \begin{bmatrix} k_{L_1} & 0 & . & 0 \\ 0 & k_{L_2} & . & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & . & 0 \end{bmatrix} \qquad [K_{NL}] = \begin{bmatrix} k_{NL_1} & 0 & . & 0 \\ 0 & k_{NL_2} & . & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & . & 0 \end{bmatrix} \tag{5.5}$$

5.2 ORTHONORMAL TRANSFORMATION

The Markov vector approach extended to nonlinear multi-degree-of-freedom systems (Nigam, 1983) is adopted for the solution of equations (5.1). Equations of motion, (5.1), with damping and force F_i set to zero, are solved for eigenvalues p_i^2, $i = 1,2,...,(n+2)$ and orthonormal modal matrix $[U]$, such that

$$\begin{aligned} &[U]^T[K][U] = [p^2] \\ &[U]^T[M][U] = [I] \end{aligned} \tag{5.6}$$

where $[p^2]$ is the diagonal eigenvalue matrix, while $[I]$ is an identity matrix.

Application of coordinate transformation

$$\{X\} = [U]\{\eta\} \tag{5.7}$$

and premultiplication by $[U]^T$, the equations of motion, (5.1), yield,

$$\ddot{\eta}_i + \beta_i \dot{\eta}_i + (1/M_i)\partial V(\eta_1, \eta_2, \cdots, \eta_{(n+2)})/\partial\eta_i = q_i \qquad i = 1,2,\ldots,n+2 \tag{5.8}$$

M_i is the modal mass. Modal damping matrix $[\beta]$ and the generalized force vector $\{q\}$ are

$$[\beta] = [U]^T[\alpha][U]$$

$$\{q\} = [U]^T\{F\} \tag{5.9}$$

and the potential energy, in generalized coordinates, can be expressed as

$$V(\eta_1, \eta_2, \cdots, \eta_{(n+2)}) = (1/2)\{\eta\}^T [U]^T ([K] + [K_L])[U]\{\eta\}$$
$$+\{g([U]\{\eta\})\}^T [K_{NL}]\{g([U]\{\eta\})\} \qquad (5.10)$$

The approach to obtaining the response of the system is simplified with the engineering assumption that the random excitation to the system is such that the generalized forces, q_i , in equations (5.8) can be treated as ideal white noise. Treating the excitation of equation (5.8) as uncorrelated Gaussian, white random forces with the following properties

$$E[q_i(t)] = 0$$
$$E[q_i(t_1)q_i(t_2)] = 2\pi\phi_i\delta(\mathrm{t}_2 - \mathrm{t}_1) \qquad (5.11)$$

where ϕ_i denotes the excitation intensity factor and $\delta(t_2 - t_1)$ is a Direc delta function.

5.3 F-P-K EQUATION

Rewriting equations (5.8) in state space form as

$$\partial\eta_i / \partial t = \dot{\eta}_i$$
$$\partial\dot{\eta}_i / \partial t = q_i - \beta_{ii}\dot{\eta}_i - (1/M_i)(\partial V / \partial\eta_i) \qquad i = 1, 2, \ldots, n+2 \qquad (5.12)$$

the drift and diffusion coefficients (refer equations (3.5 - 3.7)), for equations (5.12) can be written as

$$a_i = \dot{\eta}_i$$
$$b_j = q_j - \beta_{jj}\dot{\eta}_j - (1/M_j)[\partial V(\eta_1, \eta_2, \ldots, \eta_{n+2}) / \partial\eta_j] \qquad \begin{matrix} i = 1, 2, \ldots, n+2 \\ j = 1, 2, \ldots, n+2 \end{matrix} \qquad (5.13)$$

and

$$c_{ii} = 0$$
$$\mathrm{d}_{jj} = 2\pi\phi_j \qquad \begin{matrix} i = 1, 2, \ldots, n+2 \\ j = 1, 2, \ldots, n+2 \end{matrix} \qquad (5.14)$$
$$e_{ij} = 0$$

With the help of the above, the joint probability density function, $p(\eta_1, \eta_2, \cdots, \eta_{(n+2)}, \dot{\eta}_1, \dot{\eta}_2, \cdots, \dot{\eta}_{(n+2)})$, for the motion governed by equations (5.14) can be described by the Fokker-Planck equation

$$\sum_{i=1}^{n+2}\left[-\dot{\eta}_i \frac{\partial p}{\partial \eta_i} - \frac{1}{M_i}\frac{\partial V}{\partial \eta_i}\frac{\partial p}{\partial \dot{\eta}_i} + \frac{\partial}{\partial \dot{\eta}_i}\left(\beta_{ii}\dot{\eta}_i p + \pi\phi_i \frac{\partial p}{\partial \dot{\eta}_i}\right)\right] = \frac{\partial p}{\partial t} \qquad (5.15)$$

5.4 RESPONSE

For a stationary case equation (5.15) reduces to

$$\sum_{i=1}^{n+2}\left[-\dot{\eta}_i \frac{\partial p}{\partial \eta_i} - \frac{1}{M_i}\frac{\partial V}{\partial \eta_i}\frac{\partial p}{\partial \dot{\eta}_i} + \frac{\partial}{\partial \dot{\eta}_i}\left(\beta_{ii}\dot{\eta}_i p + \pi\phi_i \frac{\partial p}{\partial \dot{\eta}_i}\right)\right] = 0 \qquad (5.16)$$

With the assumption

$\beta_{11} / M_1\phi_1 = \beta_{22} / M_2\phi_2 = \ldots\ldots \beta_{(n+2)(n+2)} / M_{(n+2)}\phi_{(n+2)} = \gamma$

equation (5.16) can be solved to obtain the joint probability density of displacements and velocities in terms of the transformed coordinate system as

$$p(\eta_1, \eta_2, \cdots, \eta_{(n+2)}, \dot{\eta}_1, \dot{\eta}_2, \cdots, \dot{\eta}_{(n+2)}) = c\exp\left[-\frac{\gamma}{\pi}\left\{\frac{1}{2}\{\dot{\eta}\}^T[M]\{\dot{\eta}\} + V(\eta_1, \eta_2, \cdots, \eta_{(n+2)})\right\}\right] \qquad (5.17)$$

Performing the inverse orthonormal transformation and noting that the term

$(1/2)\{\dot{\eta}\}^T[M]\{\dot{\eta}\} + V(\eta_1, \eta_2, \cdots, \eta_{(n+2)})$

in the bracket on the right hand side of equation (5.17) represents the total energy of the system and that γ is a constant, the joint probability density of displacements and velocities in the original set of coordinates is

$$p(x_1,x_2,\cdots,x_{(n+2)},\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_{(n+2)}) = c\exp\left[-\frac{\gamma}{\pi}\left\{\frac{1}{2}\{\dot{X}\}^T[M]\{\dot{X}\} + V(x_1,x_2,\cdots,x_{(n+2)})\right\}\right] \tag{5.18}$$

The joint probability functions $p(x_1,x_2,\cdots,x_{(n+2)})$ and $p(\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_{(n+2)})$ are obtained fr... equation (5.18) as (Roberts and Spanos, 1990)

$$\begin{aligned} p(x_1,x_2,\cdots,x_{(n+2)}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(x_1,x_2,\cdots,x_{(n+2)},\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_{(n+2)})d\dot{x}_1 d\dot{x}_2\cdots d\dot{x}_{(n+2)} \\ &= c_1\exp\left[-\frac{\gamma}{\pi}\left\{V(x_1,x_2,\cdots,x_{(n+2)})\right\}\right] \end{aligned}$$

with

$$c_1^{-1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[-\frac{\gamma}{\pi}\left\{V(x_1,x_2,\cdots,x_{(n+2)})\right\}\right]dx_1 dx_2\cdots dx_{(n+2)} \tag{5.19}$$

Also

$$\begin{aligned} p(\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_{(n+2)}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(x_1,x_2,\cdots,x_{(n+2)},\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_{(n+2)})dx_1 dx_2\cdots dx_{(n+2)} \\ &= \left[\frac{1}{\pi}\sqrt{\frac{\gamma}{2}}\right]^{(n+2)}\sqrt{m_1 m_2\cdots m_{(n+2)}}\exp\left[-\frac{\gamma}{\pi}\left\{\frac{1}{2}\{\dot{X}\}^T[M]\{\dot{X}\}\right\}\right] \end{aligned} \tag{5.20}$$

The joint probability function $p(x_1,x_2)$ is obtained as

$$\begin{aligned} p(x_1,x_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(x_1,x_2,\cdots,x_{(n+2)})dx_3 dx_4\cdots dx_{(n+2)} \\ &= c_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[-\frac{\gamma}{\pi}V(x_1,x_2,\cdots,x_{(n+2)})\right]dx_3 dx_4\cdots dx_{(n+2)} \\ &= c_2\exp\left[-\frac{\gamma}{\pi}\left\{\frac{1}{2}(M_{\kappa 22}/M_{\kappa 1122})x_1^2 + \frac{1}{2}(M_{\kappa 11}/M_{\kappa 1122})x_2^2 + (M_{\kappa 12}/M_{\kappa 1122})x_1 x_2 \right.\right. \\ &\qquad \left.\left. + k_{NL_1}g^2(x_1) + k_{NL_2}g^2(x_2)\right\}\right] \end{aligned}$$

with

$$c_2 = \left[\pi\sqrt{\frac{2}{\gamma}}\right]^n \sqrt{\frac{1}{M_{\kappa 1122}}} \Bigg/ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[-\frac{\gamma}{\pi}V(x_1,x_2,\cdots,x_{(n+2)})\right]dx_3dx_4\cdots dx_{(n+2)}$$

$$[\kappa]=[K]+[K_L] \tag{5.21}$$

and M_κ denoting the minors of the matrix $[\kappa]$.

The probability density function $p(\dot{x}_1)$ and $p(\dot{x}_2)$ are

$$\begin{aligned} p(\dot{x}_1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_{(n+2)})d\dot{x}_2 d\dot{x}_3\cdots d\dot{x}_{(n+2)} \\ &= \left[\frac{1}{\pi}\sqrt{\frac{m_1\gamma}{2}}\right]\exp\left[-\frac{\gamma}{\pi}\left\{\frac{1}{2}m_1\dot{x}_1^2\right\}\right] \end{aligned} \tag{5.22}$$

$$\begin{aligned} p(\dot{x}_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_{(n+2)})d\dot{x}_1 d\dot{x}_3\cdots d\dot{x}_{(n+2)} \\ &= \left[\frac{1}{\pi}\sqrt{\frac{m_2\gamma}{2}}\right]\exp\left[-\frac{\gamma}{\pi}\left\{\frac{1}{2}m_2\dot{x}_2^2\right\}\right] \end{aligned} \tag{5.23}$$

The variances of velocity responses $\dot{x}_1$ and $\dot{x}_2$ are obtained as

$$\begin{aligned} \sigma_{\dot{x}_1}^2 &= \int_{-\infty}^{\infty}\dot{x}_1^2\,p(\dot{x}_1)d\dot{x}_1 \\ &= \pi/m_1\gamma \end{aligned} \tag{5.24}$$

$$\begin{aligned} \sigma_{\dot{x}_2}^2 &= \int_{-\infty}^{\infty}\dot{x}_2^2\,p(\dot{x}_2)d\dot{x}_2 \\ &= \pi/m_2\gamma \end{aligned} \tag{5.25}$$

Combining equations (5.24) and (5.25)

$$\sigma_{\dot{x}_1}\sigma_{\dot{x}_2} = (\pi/\gamma)(1/\sqrt{m_1m_2}) \tag{5.26}$$

the joint probability density function for the displacement responses x_1 and x_2, from equations (5.21) and (5.26), can be written as

$$p(x_1, x_2) = c_2 \exp\left[-\frac{1}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}\sqrt{m_1 m_2}} \left\{ \frac{1}{2}(M_{\kappa 22} / M_{\kappa 1122})x_1^2 + \frac{1}{2}(M_{\kappa 11} / M_{\kappa 1122})x_2^2 \right.\right.$$

$$\left.\left. + (M_{\kappa 12} / M_{\kappa 1122})x_1 x_2 + k_{NL_1} g^2(x_1) + k_{NL_2} g^2(x_2) \right\}\right]$$

with

$$c_2 = \sqrt{\frac{\left[2\pi\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}\sqrt{m_1 m_2}\right]^n}{M_{\kappa 1122}}} \Bigg/ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left[-\frac{V(x_1, x_2, \cdots, x_{(n+2)})}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}\sqrt{m_1 m_2}} \right] dx_3 dx_4 \cdots dx_{(n+2)} \tag{5.27}$$

5.5 EXTRACTION OF BEARING PARAMETERS

Based on the above analysis, the bearing parameters, namely, the linear stiffness parameters k_{l_1}, k_{l_2}, the nonlinear stiffness parameters k_{NL_1}, k_{NL_2} and the bearing masses m_1 and m_2, are extracted from experimentally obtained random response. These parameters are obtained for both, the vertical and horizontal directions. The problem formulation, in the horizontal direction, remains identical to that in the vertical direction.

The joint probability density function $p(x_1, x_2)$ for a set of displacements (x_{1_i}, x_{2_j}) $(x_{1_{(i+1)}}, x_{2(j+1)})$, $(x_{1_{(i+1)}} \rangle x_{1_i}$ and $x_{2_{(j+1)}} \rangle x_{2_j})$ from equation (5.27) are

$$p(x_{1_i}, x_{2_j}) = c_2 \exp\left[-\frac{1}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}\sqrt{m_1 m_2}} \left\{ \frac{1}{2}(M_{\kappa 22} / M_{\kappa 1122})x_{1_i}^2 + \frac{1}{2}(M_{\kappa 11} / M_{\kappa 1122})x_{2_j}^2 \right.\right.$$

$$\left.\left. + (M_{\kappa 12} / M_{\kappa 1122})x_{1_i} x_{2_j} + k_{NL_1} g^2(x_{1_i}) + k_{NL_2} g^2(x_{2_j}) \right\}\right] \tag{5.28}$$

$$p(x_{1_{(i+1)}}, x_{2_{(j+1)}}) = c_2 \exp\left[-\frac{1}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}\sqrt{m_1 m_2}} \left\{ \frac{1}{2}(M_{\kappa 22} / M_{\kappa 1122})x_{1_{(i-1)}}^2 + \frac{1}{2}(M_{\kappa 11} / M_{\kappa 1122})x_{2_{(j-1)}}^2 \right.\right.$$

$$\left.\left. + (M_{\kappa 12} / M_{\kappa 1122})x_{1_{(i-1)}} x_{2_{(j-1)}} + k_{NL_1} g^2(x_{1_{(i+1)}}) + k_{NL_2} g^2(x_{2_{(j-1)}}) \right\}\right] \tag{5.29}$$

Defining $\qquad \Delta x_{1_i} = x_{1_{(i+1)}} - x_{1_i}$; $\qquad \Delta x_{2_j} = x_{2_{(j+1)}} - x_{2_j}$

for small Δx_{1_i} and Δx_{2_j}, one can write

$$p(x_{1_{(i+1)}}, x_{2_{(j+1)}}) = c_2 \exp\Bigg[-\frac{1}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}\sqrt{m_1 m_2}} \Big\{ \frac{1}{2}(M_{\kappa 22} / M_{\kappa 1122}) x_{1_i}^2 + \frac{1}{2}(M_{\kappa 11} / M_{\kappa 1122}) x_{2_j}^2$$

$$+ (M_{\kappa 12} / M_{\kappa 1122}) x_{1_i} x_{2_j}$$

$$+ k_{NL_1} g^2(x_{1_i}) + k_{NL_2} g^2(x_{2_j}) \Big\} \Bigg]$$

$$\exp\Bigg[\frac{-1}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}\sqrt{m_1 m_2}} \Big\{ (M_{\kappa 22} / M_{\kappa 1122}) x_{1_i} \Delta x_{1_i} + (M_{\kappa 11} / M_{\kappa 1122}) x_{2_j} \Delta x_{2_j}$$

$$+ (M_{\kappa 12} / M_{\kappa 1122})(x_{1_i} \Delta x_{2_j} + x_{2_j} \Delta x_{1_i})$$

$$+ k_{NL_1} G(x_{1_i}) \Delta x_{1_i} + k_{NL_2} G(x_{2_j}) \Delta x_{2_j} \Big\} \Bigg] \tag{5.30}$$

Combining equations (5.30) and (5.28) gives

$$p(x_{1_{(i+1)}}, x_{2_{(j+1)}}) = p(x_{1_i}, x_{2_j}) \exp\Bigg[\frac{-1}{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}\sqrt{m_1 m_2}} \Big\{ (M_{\kappa 22} / M_{\kappa 1122}) x_{1_i} \Delta x_{1_i}$$

$$+ (M_{\kappa 11} / M_{\kappa 1122}) x_{2_j} \Delta x_{2_j}$$

$$+ (M_{\kappa 12} / M_{\kappa 1122})(x_{1_i} \Delta x_{2_j} + x_{2_j} \Delta x_{1_i}) \tag{5.31}$$

$$+ k_{NL_1} G(x_{1_i}) \Delta x_{1_i} + k_{NL_2} G(x_{2_j}) \Delta x_{2_j} \Big\} \Bigg]$$

For N values of each displacement, $x_{1_1}, x_{1_2}, \ldots\ldots\ldots, x_{1_N}$, and $x_{2_1}, x_{2_2}, \ldots\ldots\ldots, x_{2_N}$, equation (5.31) can be expressed as a set of $(N-1)^2$ linear simultaneous algebraic equations, as,

$$\left[\frac{\sigma_{\dot{x}_1}\sigma_{\dot{x}_2}}{\Delta x_{1_i}\Delta x_{2_j}}\ln\left\{\frac{p(x_{1_i},x_{2_j})}{p(x_{1_{(i+1)}},x_{2_{(j+1)}})}\right\}\right]\left\{\frac{\sqrt{m_1 m_2}}{k_{L_1}}\right\}-\left[\frac{G(x_{1_i})}{\Delta x_{2_j}}\right]\left\{\frac{k_{NL_1}}{k_{L_1}}\right\}-\left[\frac{G(x_{2_j})}{\Delta x_{1_i}}\right]\left\{\frac{k_{NL_2}}{k_{L_1}}\right\}$$

$$-\left[\left(M_{\kappa 22}/M_{\kappa 1122}\right)\left(\frac{x_{1_i}}{\Delta x_{2_j}}\right)+\left(M_{\kappa 11}/M_{\kappa 1122}\right)\left(\frac{x_{2_i}}{\Delta x_{1_i}}\right)+\left(M_{\kappa 12}/M_{\kappa 1122}\right)\left(\frac{x_{1_i}}{\Delta x_{1_i}}+\frac{x_{2_i}}{\Delta x_{2_j}}\right)\right]\left\{\frac{1}{k_{L_1}}\right\}$$

$$+\left(\frac{x_{2_j}}{\Delta x_{1_i}}\right)\left\{1-\frac{k_{L_2}}{k_{L_1}}\right\}$$

$$=\left(\frac{x_{1_i}}{\Delta x_{2_j}}+\frac{x_{2_j}}{\Delta x_{1_i}}\right) \qquad i=1,2,\dots\dots\dots,(N-1) \qquad j=1,2,\dots\dots\dots,(N-1)$$

(5.32)

Equations (5.32) are solved for k_{L_1}, k_{L_2}, k_{NL_1}, k_{NL_2}, and $\sqrt{m_1 m_2}$, using the least square fit technique.

5.6 EXPERIMENTATION

Two discs are mounted at equidistances on the shaft of the laboratory rotor, as shown in Figures 5.2 and 5.3. The shaft remains supported in the same bearings that were used for illustration in the previous chapters. The shaft is driven through a flexible coupling by a motor and the vibration signals are picked up (after balancing the rotor) in both, the vertical and horizontal directions, by accelerometers mounted on both of the bearing housing.

The nonlinear spring force provided by the rolling element bearings is taken to be cubic in nature (Ragulskis, et al., 1974) i.e. $G(x) = x^3$. The stiffness-matrix of the shaft model of the rig is obtained as (Childs, 1993)

$$[K]=\frac{12EI}{L^3}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ -1 & 0 & 2 & -1\\ 0 & -1 & -1 & 2\end{bmatrix}$$

$EI = 1.03 \times 10^8$ N-mm^2 $\qquad L = 165.0$ mm. (5.33)

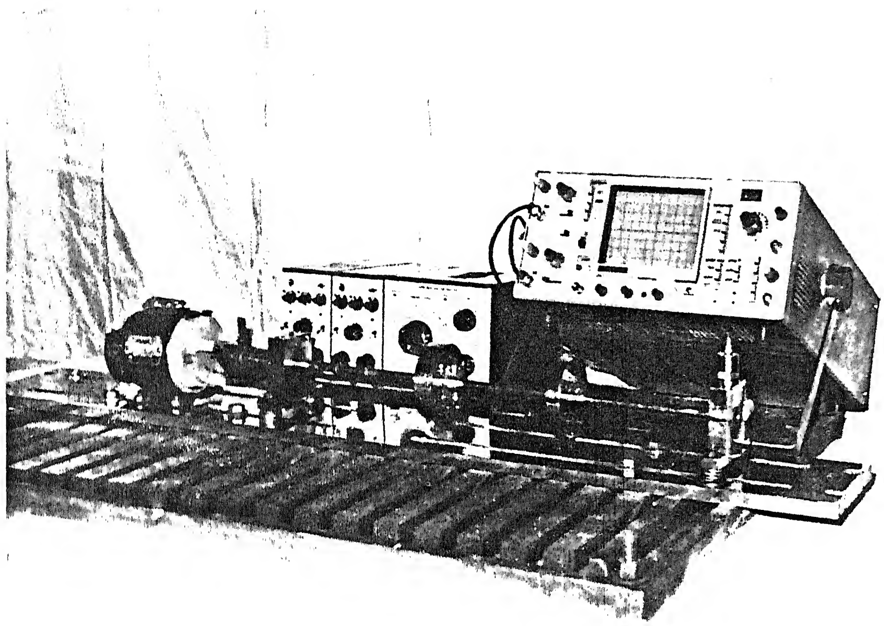


Figure 5.2 Two disc rotor bearing set-up

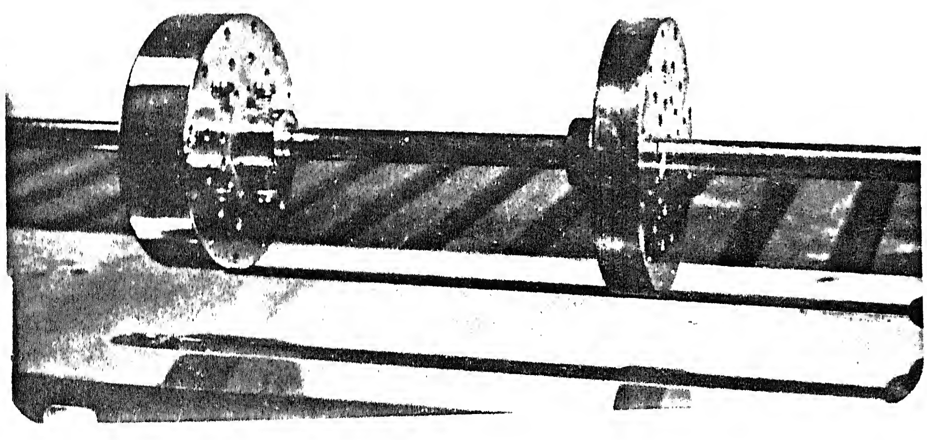


Figure 5.3 Close view of discs

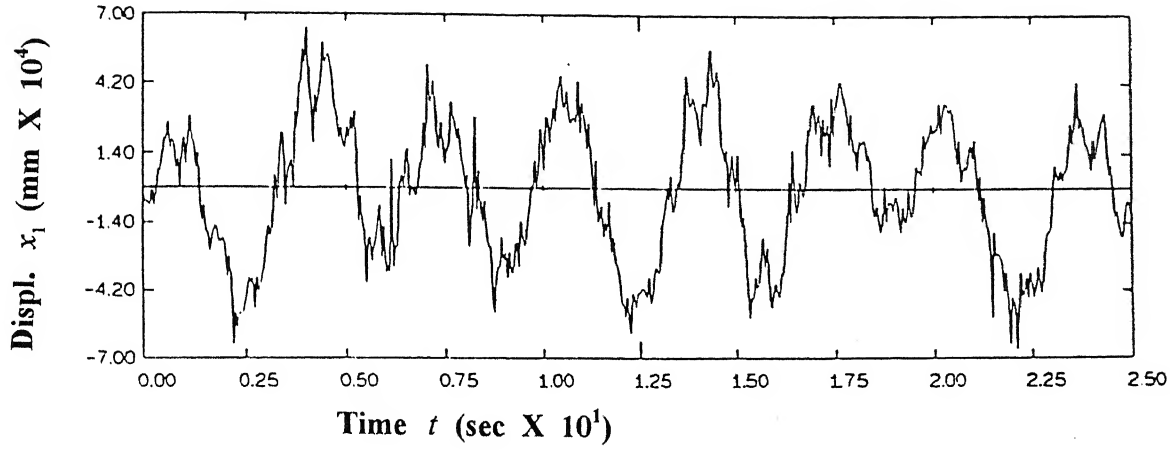


Figure 5.4 Displacement signal in the vertical direction at bearing 1

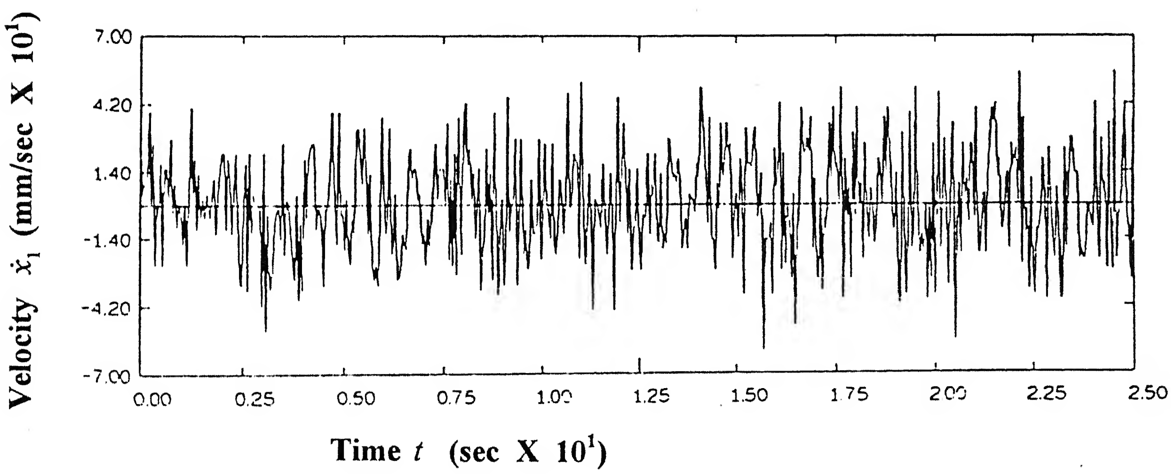


Figure 5.5 Velocity signal in the vertical direction at bearing 1

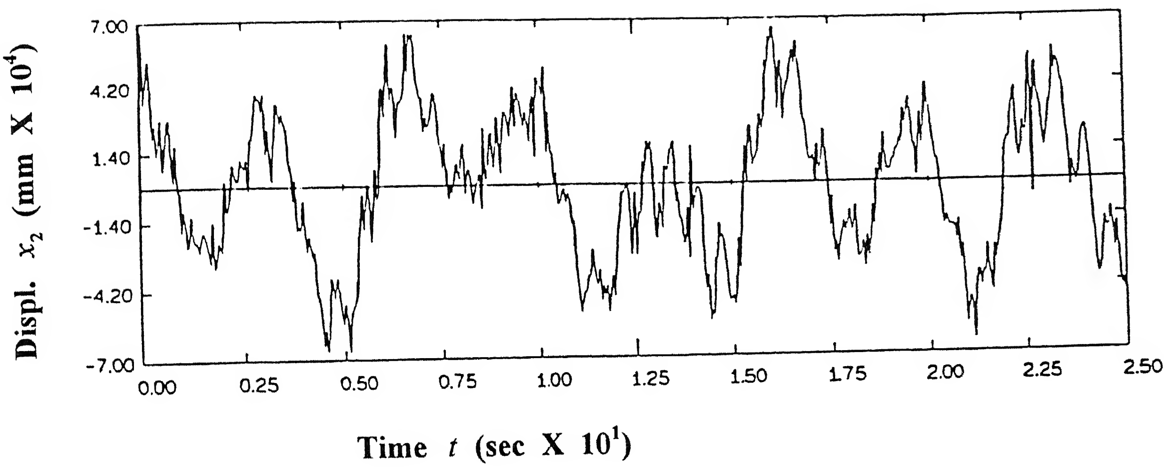


Figure 5.6 Displacement signal in the vertical direction at bearing 2

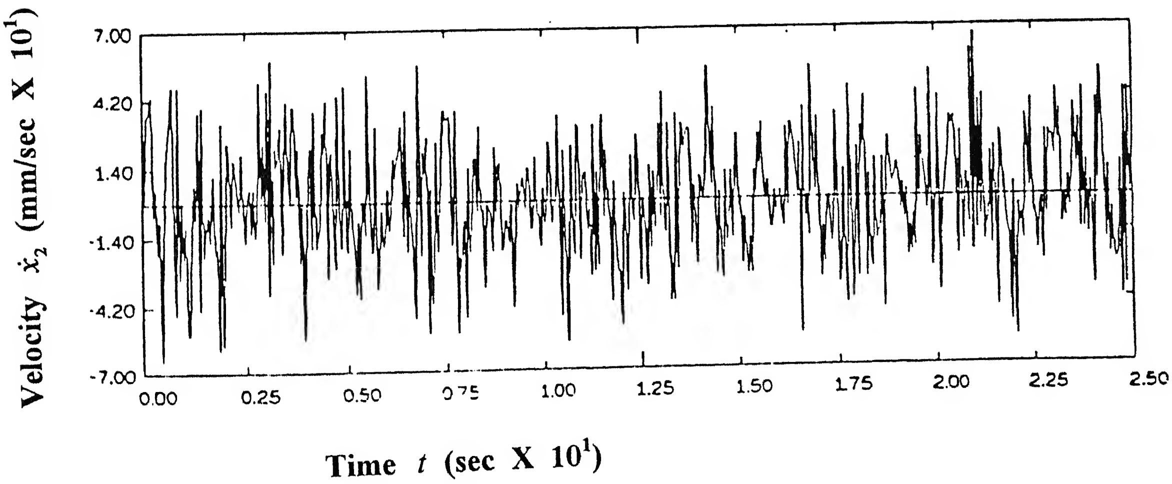


Figure 5.7 Velocity signal in the vertical direction at bearing 2

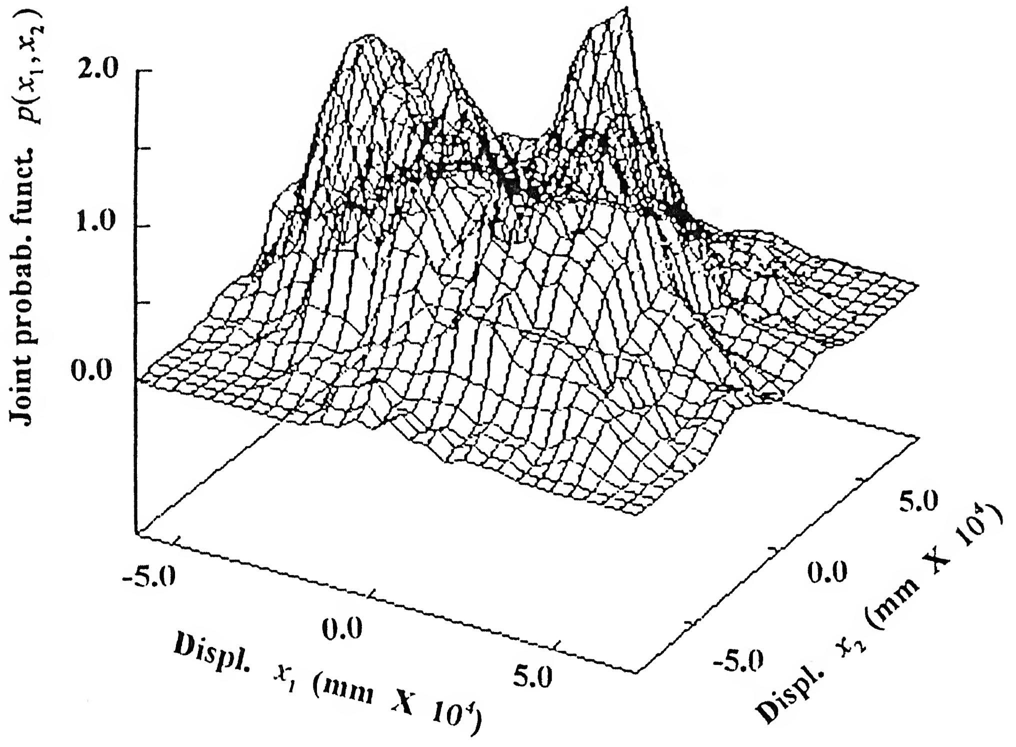


Figure 5.8 Joint probability density distribution of vertical displacements at bearing 1 and bearing 2

The disc masses are

$m_3 = 0.515$ Kg. $\qquad\qquad$ $m_4 = 0.755$ Kg.

(The procedure, however, does not require a knowledge of the disc masses.)

Typical experimentally obtained displacement and velocity signals ($x_1, x_2, \dot{x}_1$ and $\dot{x}_2$), in the vertical direction, picked up by the accelerometer are given in Figures 5.4, 5.5, 5.6, and 5.7. The joint probability function, $p(x_1, x_2)$ and variances, $\sigma^2_{\dot{x}_1}$ and $\sigma^2_{\dot{x}_2}$ are computed from the measured responses. The joint probability density function, $p(x_1, x_2)$, of the displacements is shown in Figure 5.8. The bearing parameters estimated from equations (5.32) are given in Table 5.1.

Table 5.1 Estimated bearing stiffness and mass parameters

Parameters	Vertical	Horizontal
k_{L_1} **(N/mm)**	1.04×10^4	0.87×10^4
k_{NL_1} **(N/mm^3)**	-5.10×10^{10}	-3.50×10^{10}
k_{L_2} **(N/mm)**	1.04×10^4	0.86×10^4
k_{NL_2} **(N/mm^3)**	$- 3.62 \times 10^{10}$	$- 2.19 \times 10^{10}$
$\sqrt{m_1 m_2}$ **(Kg)**	0.21	0.20

(Refer to Appendix C for statistical Error Table)

5.7 MONTE CARLO SIMULATION

Monte Carlo simulation is employed to check the accuracies involved in making the engineering assumptions in the algorithm. The above estimated values of $k_{L_1}, k_{L_2}, k_{NL_1}, k_{NL_2}$ and $\sqrt{m_1 m_2}$ are fed into equation (5.1). Broad band excitation forces, $F_1(t)$ and $F_2(t)$ with zero mean and Gaussian probability distribution, as described in Figures 5.9, 5.10, 5.11 and 5.12, are simulated on the computer and these are also fed into equation (5.1). Fourth order Runga-Kutta technique is

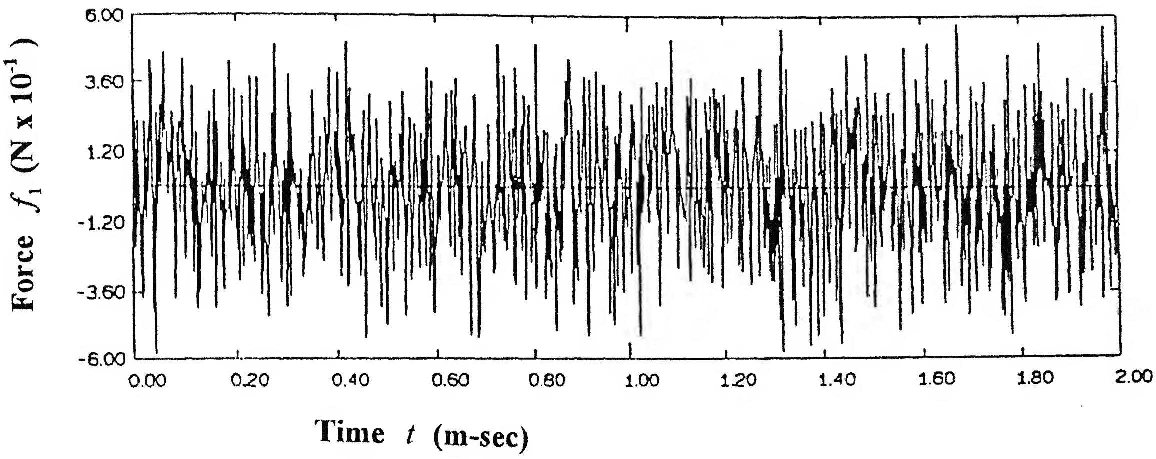


Figure 5.9 **Simulated random force at bearing 1**

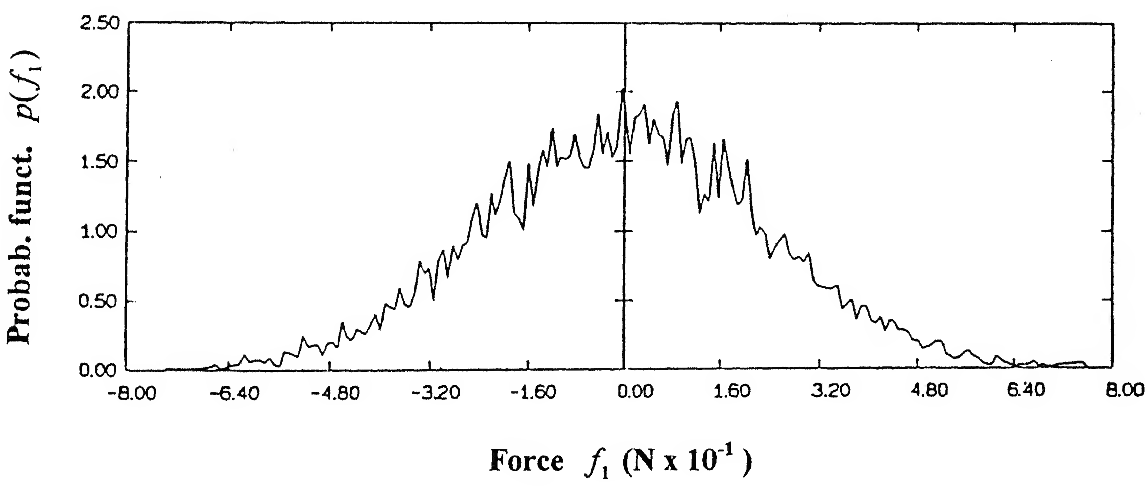


Figure 5.10 **Probability density distribution of simulated force at bearing 1**

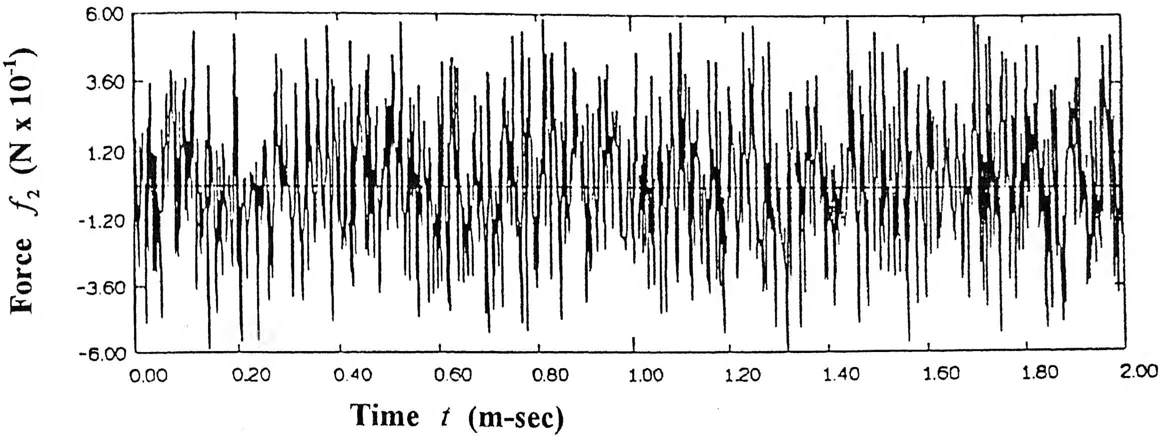


Figure 5.11 Simulated random force at bearing 2

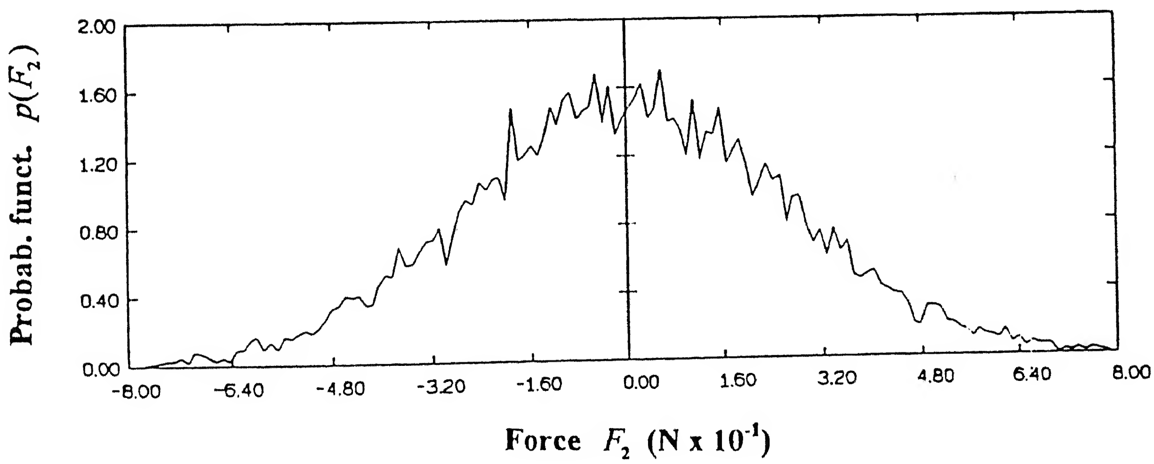


Figure 5.12 Probability density distribution of simulated force at bearing 2

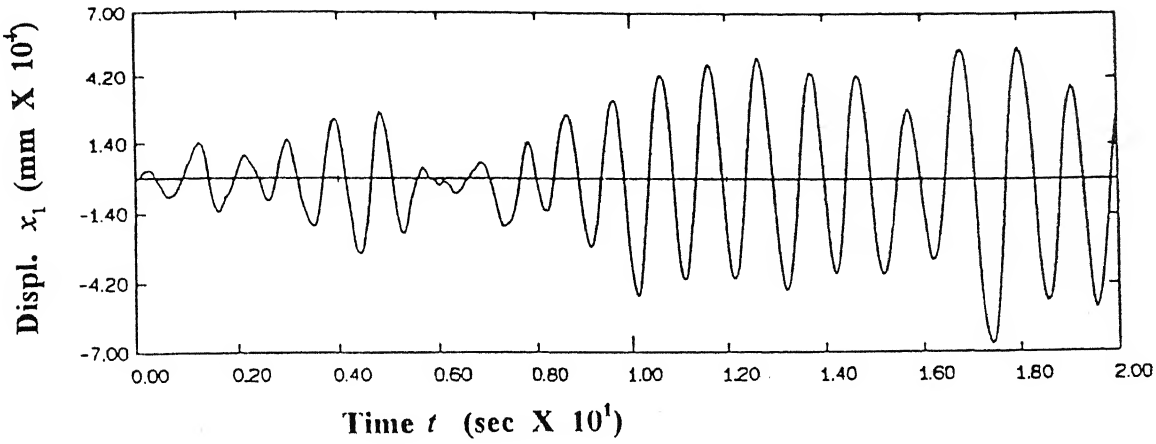


Figure 5.13 **Simulated displacement signal at bearing 1**

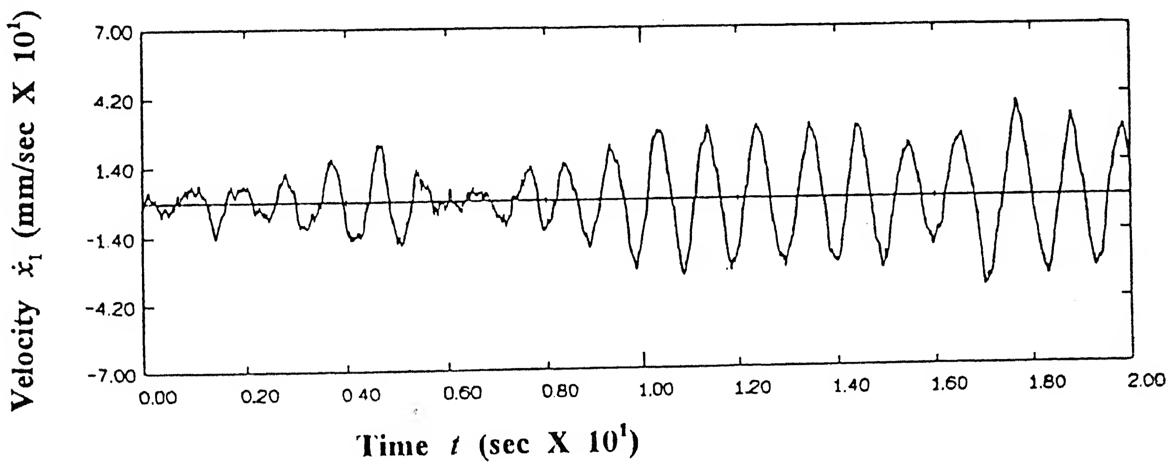


Figure 5.14 **Simulated velocity signal at bearing 1**

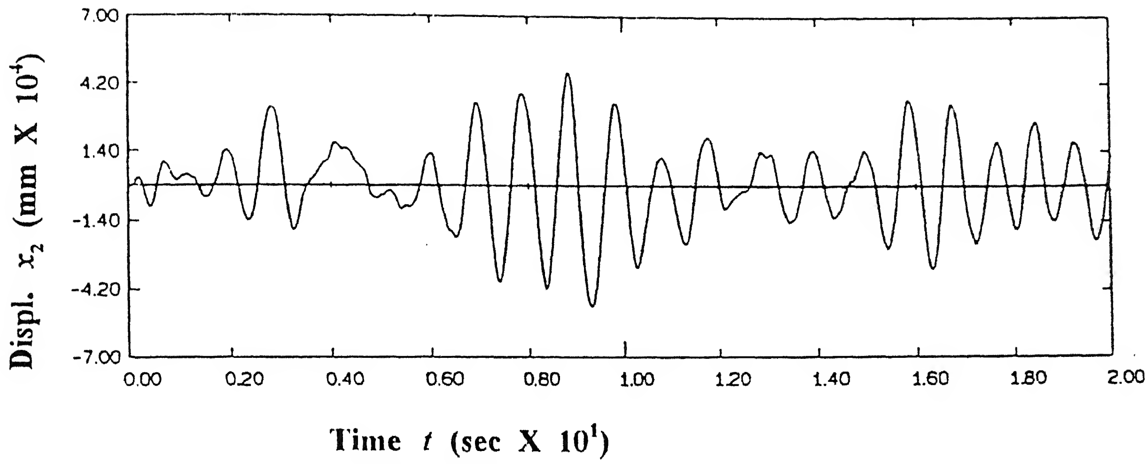


Figure 5.15 Simulated displacement signal at bearing 2

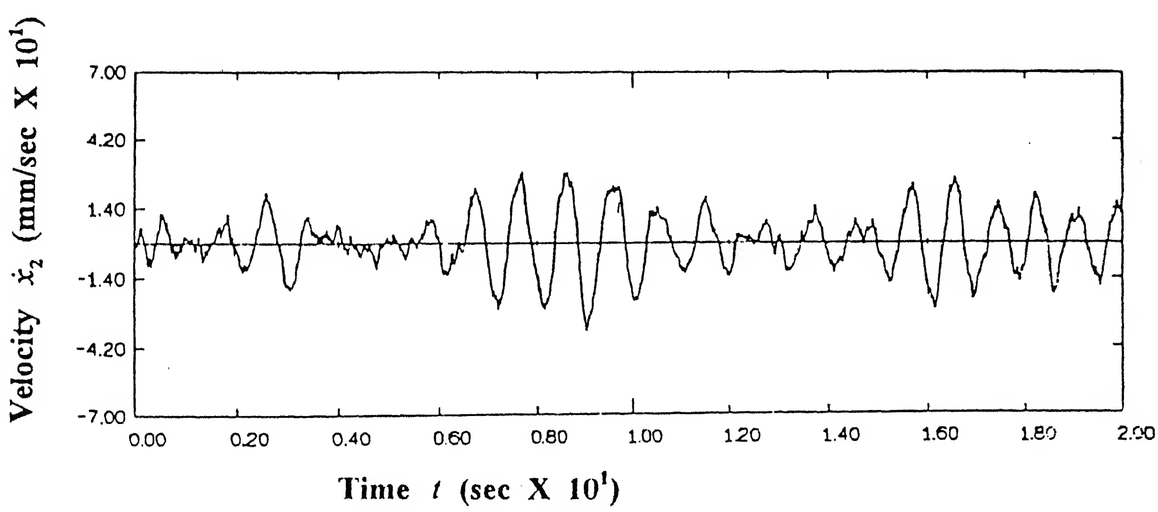


Figure 5.16 Simulated velocity signal at bearing 2

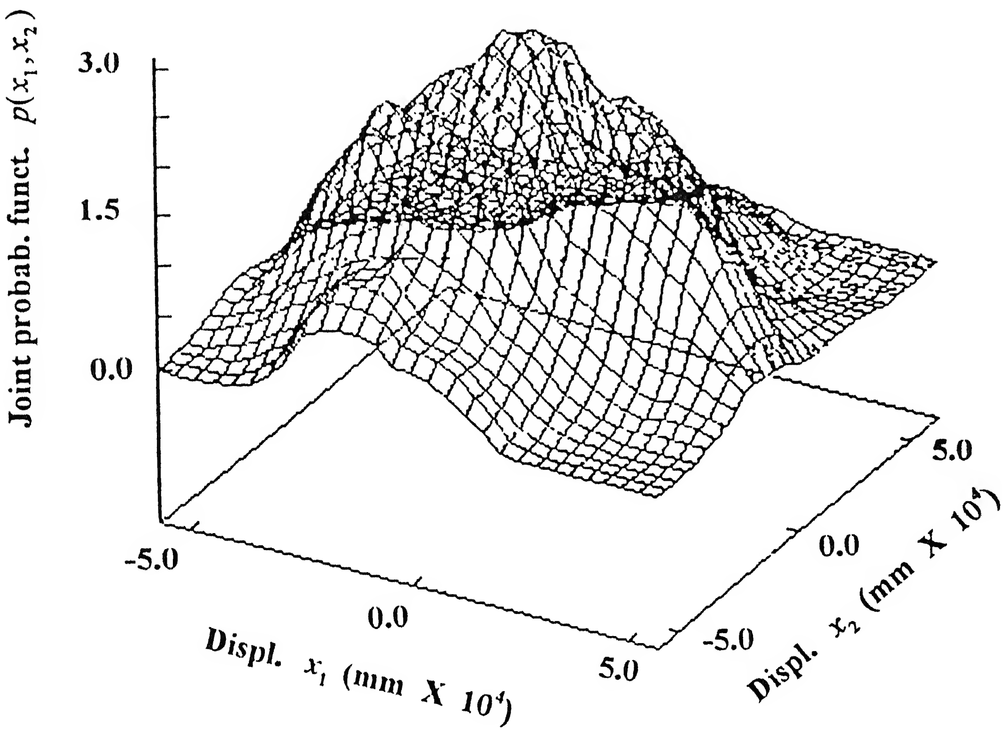


Figure 5.17 Joint probability density distribution of simulated vertical displacements at bearing 1 and bearing 2

then employed to numerically compute the displacement and velocity responses, $x_1, x_2, \dot{x}_1$ and $\dot{x}_2$. The displacement and velocity responses at the two bearings, thus simulated are given in Figures 5.13, 5.14, 5.15 and 5.16, for the vertical direction. The joint probability distribution of the simulated vertical displacements is shown in Figure 5.17. The simulated response is now fed into equation (5.32) to obtain the values of $k_{L_1}, k_{L_2}, k_{NL_1}, k_{NL_2}$ and $\sqrt{m_1 m_2}$. A similar exercise is carried out to obtain the parameters in the horizontal direction. These values are listed in Table 5.2.

Table 5.2 Estimated and simulated bearing stiffness and mass parameters

	Vertical		Horizontal	
Parameters	**Experimental**	**Simulated**	**Experimental**	**Simulated**
k_{L_1} (N/mm)	1.04 x 10^4	1.49 x 10^4	0.87 x 10^4	0.99 x 10^4
k_{NL_1} (N/mm^3)	-5.10 x 10^{10}	-8.48 x 10^{10}	-3.50 x 10^{10}	-4.81 x 10^{10}
k_{L_2} (N/mm)	1.04 x 10^4	1.48 x 10^4	0.86 x 10^4	0.98 x 10^4
k_{NL_2} (N/mm^3)	- 3.62 x 10^{10}	-7.83 x 10^{10}	- 2.19 x 10^{10}	-3.23 x 10^{10}
$\sqrt{m_1 m_2}$ (Kg)	0.21	0.26	0.20	0.24

The fairly good agreement between the values of the bearing stiffness parameters, $k_{L_1}, k_{L_2}, k_{NL_1}, k_{NL_2}$ and $\sqrt{m_1 m_2}$, obtained by processing the experimental data and those from the Monte Carlo simulation, indicate the correctness of the experimental and algebraic exercises. It should be noted that the simulated values of the bearing stiffness parameters are obtained for an ideal white noise excitation, while the experimental ones are obtained by processing the actual response of the system, where the unknown excitation was idealized as white noise. It also needs to be pointed out that the values of the damping parameters α_{ij}, are not required for the estimation procedure (equation (5.32)). Any convenient set of values of α_{ij}, can be employed in equations (5.1) for the purpose of simulation.

5.8 VALIDATION

The bearings in the laboratory rotor rig employed to illustrate the procedure, are the same, as those used for illustrations in the previous chapters. Table 5.3 makes a comparison between the stiffness parameters obtained by processing the experimental signals from the two-disc rotor with the expressions for bearing stiffnesses obtained through the analytical formulations of Ragulskis, et al. (1974) and Harris (1984), described in Chapter 3.

Table 5.3 **Estimated and theoretical (Ragulskis, et al., 1974; Harris, 1984) bearing stiffness parameters**

Preload (mm)	**Theoretical Stiffness (Radial) (N/mm)**	**Estimated Stiffness (N/mm) (at bearing 1)**	**Estimated Stiffness (N/mm) (at bearing 2)**
0.0002	$1.20 \times 10^4 - 4.01 \times 10^{10}\ x^2$		
		$0.87 \times 10^4 - 3.50 \times 10^{10}\ x^2$ **(horizontal)**	$0.86 \times 10^4 - 2.19 \times 10^{10}\ x^2$ **(horizontal)**
0.0003	$1.47 \times 10^4 - 2.18 \times 10^{10}\ x^2$		
0.0004	$1.69 \times 10^4 - 1.42 \times 10^{10}\ x^2$		
0.0005	$1.89 \times 10^4 - 1.02 \times 10^{10}\ x^2$		
		$1.04 \times 10^4 - 5.10 \times 10^{10}\ x^2$ **(vertical)**	$1.04 \times 10^4 - 3.62 \times 10^{10}\ x^2$ **(vertical)**
0.0006	$2.08 \times 10^4 - 6.09 \times 10^9\ x^2$		

The theoretical stiffness in the above table is dependent on the preloading, g, of the balls, which may be difficult to guess, for bearings which have involved wear and tear under operation. The estimated stiffnesses can be seen to lie within the preload range specified by the manufacturer. The analytical stiffness of the test bearing is also plotted in Figures 5.18 and 5.19 as a function of the radial deformation, x, for various allowable preload values, g. along with the estimated bearing

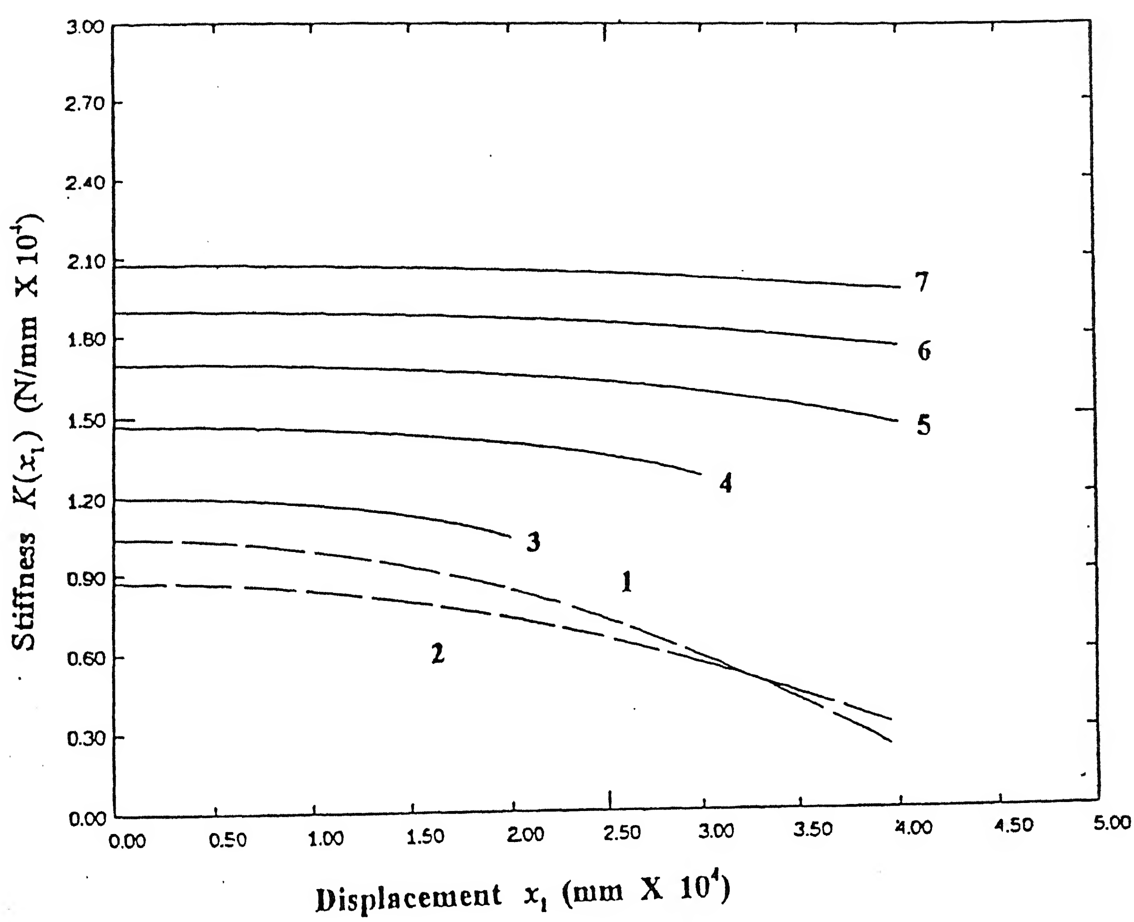


Figure 5.18 Stiffness comparison at bearing 1
1,2 - Present study (In vertical and horizontal directions respectively)
3,4,5,6,7 - Harris (1984) and Ragulskis et al. (1974) with prelaod 0.0002, 0.0003, 0.0004, 0.0005 and 0.0006 mm. respectively (In any radial direction)

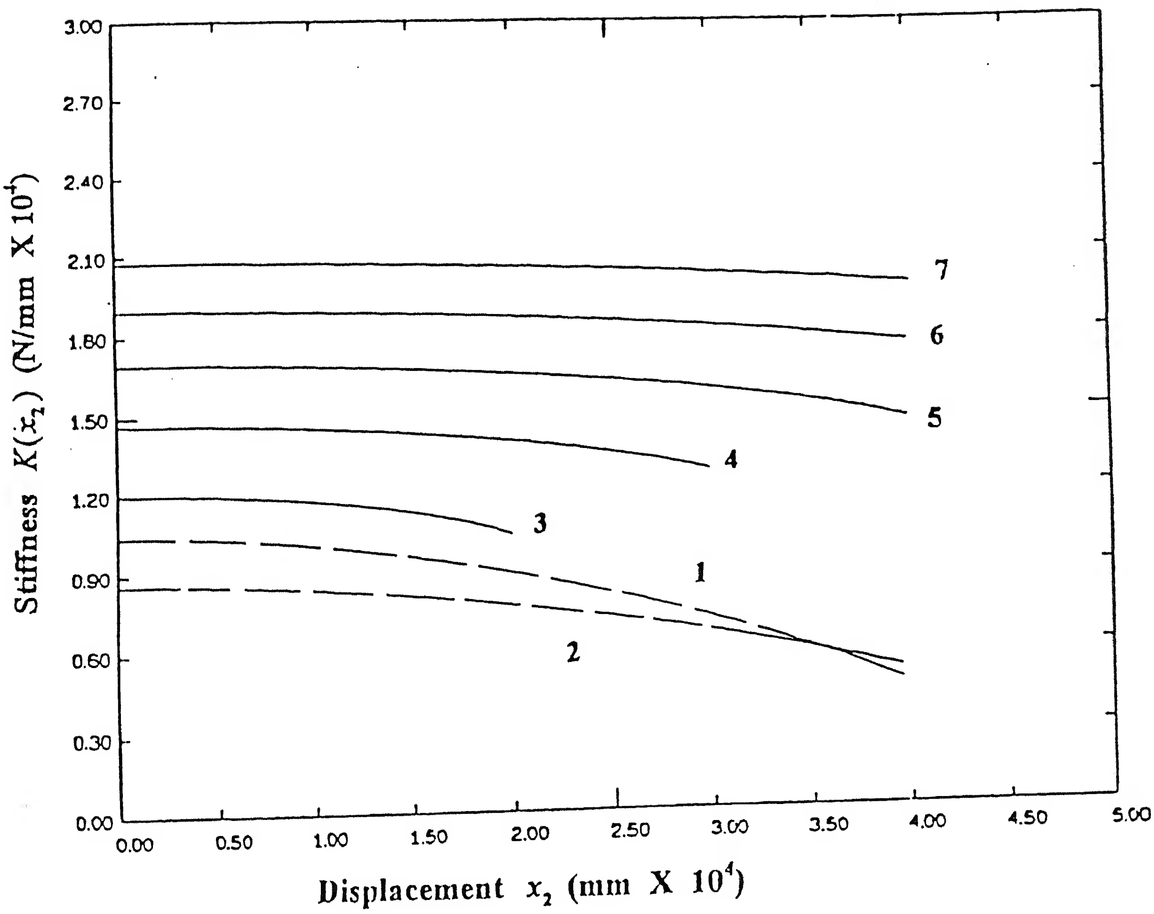


Figure 5.19 **Stiffness comparison at bearing 2**
1,2 - Present study (In vertical and horizontal directions respectively)
3,4,5,6,7 - Harris (1984) and Ragulskis et al. (1974) with prelaod 0.0002, 0.0003, 0.0004, 0.0005 and 0.0006 mm. respectively (In any radial direction)

stiffnesses. It needs to be emphasized again, that the theoretical stiffness calculations are based on formulations which analyse the bearing in isolation of the shaft.

The experimentally obtained values of $\sqrt{m_1 m_2}$, a parameter representing the effective masses at the bearing ends, are 0.21 and 0.20 (Table 5.1) in the vertical and horizontal directions respectively. If the two bearings were taken to be identical for each of the bearing ends turns out to be 0.21 Kgs in the vertical direction and 0.20 Kgs in the horizontal direction. These values look reasonable, in view of the fact that along with some contribution from the bearings themselves, a division of the mass of the shaft, which in this case is 0.306 Kgs, is seen at the two bearing ends.

5.9 REMARKS

Generalized expressions for the governing equations, orthonormal transformation, rotor response provide a compact algorithm for parameter extraction. A comparison between the bearing stiffness values obtained in this chapter, can be made with those obtained in the previous chapters, since the test bearings employed in the experimental rigs are the same. The closeness of results is an indication of the correctness of the theoretical and experimental exercises.

CHAPTER 6

PARAMETER ESTIMATION IN NON-LINEAR ROTOR -BEARING SYSTEMS WITH UNBALANCE

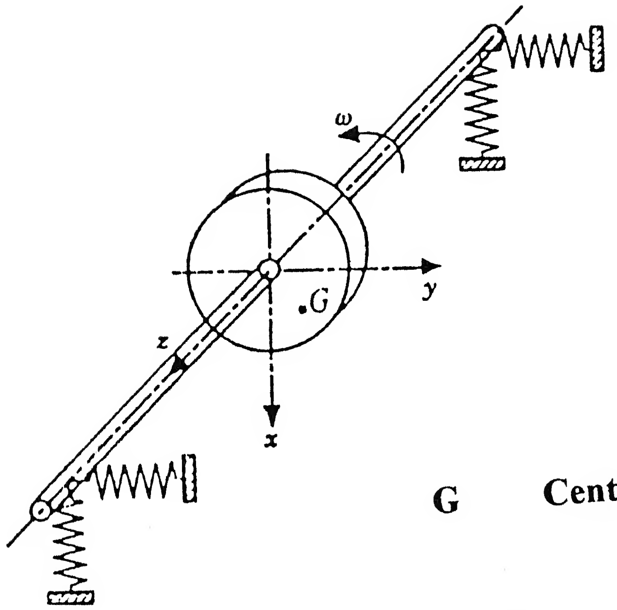
Estimation of linear and non-linear bearing stiffness parameters, for rotors with unknown unbalance, is considered in this chapter. The problem is formulated as single degree of freedom system. The case of a rotor with a rigid shaft, in nonlinear flexible bearings is investigated. The excitation to the system consists harmonic forces due to the unbalance and random forces due to arbitrary deviations, of bearing contact surfaces and subsurfaces, from their ideal design and their progressive deterioration during operation. These random forces are comparable to the harmonic excitation forces, if the unbalance is not significantly large.

The parameter estimation procedure is based on the averaging technique of Bogoliubov and Mitropolsky (1961) for deterministic non-linear systems, extended by Stratonovich (1967) for stochastic differential equations. The governing equation of motion is transformed from the rapidly varying variables, namely displacement and velocity, to variables, amplitude and phase, varying slowly with time. Stochastic averaging is done, to take into account the effect of the random excitation multiplied by a correlated term, so as to model the slowly varying amplitude as an approximate Markovian process. A second order stochastic approximation is carried out and a one-dimensional Fokker-Planck equation is derived to describe the Markovian amplitude process. The response to the Fokker-Planck equation is derived and processed for parameter estimation.

6.1 EQUATION OF MOTION

The analysis is restricted to rotors with the disc mounted on a rigid shaft, the non-linear spring force being contributed by the bearings (Figure 6.1). The damping to the system is taken to be linear and the disc carries an unbalance F_0. The governing equation for the system is written as

$$m\ddot{x} + c\dot{x} + k_L x + k_{NL} G(x) = F_0 \cos(\omega t + \theta) + \psi(t) \tag{6.1}$$



G Centre of gravity

Figure 6.1 Rotor-bearing model

In the equation (6.1), k_L and k_{NL} are the linear and non-linear stiffness parameters of the bearings and $G(x)$ can be a polynomial in x . The rotor mass is m and ω is its rotational speed. The angular location of the unbalance with respect to a reference point on the shaft is expressed by θ . $\psi(t)$ is the random excitation experienced at the bearing ends. The spring force nonlinearity in rolling element bearings is taken to be cubic in nature (Ragulskis, 1974), i.e.

$$G(x) = x^3 \tag{6.2}$$

6.2 STANDARD FORM TRANSFORMATION

The concept of averaging principle, developed by Bogoliubov and Mitropolsky (1961) for deterministic non-linear vibration, transforms the equation, involving vibrations which are rapidly varying with time, to a set of simple equations for slowly varying response coordinates. This principle, extended by Stratonovich (1967) for stochastic differential equations, has been employed to analyse the rotor-bearing system governed by equation (6.1).

Defining

$$\lambda = k_{NL} / k_L \tag{6.3}$$

and since $(1/\lambda)$ is a small quantity (rolling element bearings being highly nonlinear), equation (6.1) can be rewritten in terms of the small parameter $\varepsilon = (1/\lambda)$ as

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, \zeta(t))$$

where

$$\begin{aligned} f(x,\dot{x},\zeta(t)) &= [(F_0\lambda / m)\cos(\omega t + \theta) + \zeta(t)\lambda - 2\omega_n\lambda\xi\dot{x} - (\omega_n^2 - \omega^2)\lambda x - \omega_n^2\lambda^2 x^3] \\ \omega_n^2 &= k_L / m \\ \xi &= c / 2m\omega_n \\ \zeta(t) &= \psi(t) / m \end{aligned} \tag{6.4}$$

ε being a small quantity, the response can be taken to be harmonic in time with frequency ω with slowly varying amplitude, $A(t)$ and phase, $\varphi(t)$, i.e.

$$x(t) \approx A(t)\cos[\omega t + \varphi(t)]$$
$$\dot{x}(t) \approx -\omega A(t)\sin[\omega t + \varphi(t)] \tag{6.5}$$

Equation (6.4) can, now, be expressed as a set of standard form equations in terms of the slowly varying parameters $A(t)$ and $\varphi(t)$ as

$$A = [x^2 + (\dot{x}^2 / \omega^2)]^{1/2}$$
$$\varphi = -\arctan[\dot{x} / \omega x] - \omega t \tag{6.6}$$

or

$$\dot{A} = \varepsilon G[A, \varphi, \zeta(t)]$$
$$\dot{\varphi} = \varepsilon H[A, \varphi, \zeta(t)] \tag{6.7}$$

where

$$G[A, \varphi, \zeta(t)] = G_{av}(A, \varphi) - \{\zeta(t)\lambda / \omega\}\sin(\omega t + \varphi)$$

$$H[A, \varphi, \zeta(t)] = H_{av}(A, \varphi) - \{\zeta(t)\lambda / \omega A\}\cos(\omega t + \varphi) \tag{6.8}$$

with

$$\begin{aligned} G_{av}(A, \varphi) = {} & \{-(\omega\lambda A/2) + (\omega_n^2 \lambda A / 2\omega) + (\omega_n^2 \lambda^2 A^3 / 4\omega)\}\sin 2(\omega t + \varphi) \\ & + \{\omega_n \lambda \xi A\}\cos 2(\omega t + \varphi) - (F_0 \lambda / 2m\omega)\sin(2\omega t + \varphi + \theta) \\ & + (\omega_n^2 \lambda^2 A^3 / 8\omega)\sin 4(\omega t + \varphi) - \{(F_0 \lambda / 2m\omega)\sin(\varphi - \theta) + \omega_n \lambda \xi A\} \end{aligned}$$

$$\begin{aligned} H_{av}(A, \varphi) = {} & -(\omega_n \lambda \xi)\sin 2(\omega t + \varphi) + \{-(\omega\lambda/2) + (\omega_n^2 \lambda / 2\omega) \\ & + (\omega_n^2 \lambda^2 A^2 / 2\omega)\}\cos 2(\omega t + \varphi) - (F_0 \lambda / 2m\omega A)\cos(2\omega t + \varphi + \theta) \\ & + (\omega_n^2 \lambda^2 A^2 / 8\omega)\cos 4(\omega t + \varphi) + \{-(\omega\lambda/2) - (F_o \lambda / 2m\omega A)\cos(\varphi - \theta) \\ & + (\omega_n^2 \lambda / 2\omega) + (3\omega_n^2 \lambda^2 A^2 / 8\omega)\} \end{aligned} \tag{6.9}$$

6.3 STOCHASTIC AVERAGING

It can be seen that the right hand side of equations (6.7) contain (employing the terminology of (Stratonovich, 1967)) 'oscillatory' terms, i.e. harmonic functions of ωt, along with *randomly* 'fluctuating' terms, i.e. $-\{\zeta(t)\lambda/\omega\}\sin(\omega t+\varphi)$ and $-\{\zeta(t)\lambda/\omega A\}\cos(\omega t+\varphi)$, which contain the random force term $\zeta(t)$. However, due to the presence of ε in the equation (6.4), the parameters A and φ vary slowly with time and can be assumed to remain constant over a cycle of oscillation. The averaging process for A and φ can be carried out in two stages - by stochastic averaging and elimination of the randomly fluctuating terms involving random force term $\zeta(t)$ and then averaging over a cycle of oscillation for removal the oscillating terms involving harmonic functions of ωt.

The approach to obtaining the response of the system, including stochastic averaging, can be simplified by providing arguments similar to those in the previous chapters and treating the random excitation to the system as ideal white noise with zero mean and Gaussian distribution.

For a zero mean random excitation $\zeta(t)$, the expressions (6.7) for $\dot{A}$ and $\dot{\varphi}$ can be stochastically averaged to write the non-'fluctuating' amplitude term, $\dot{A}_{nf}$ and the non-'fluctuating' phase term, $\dot{\varphi}_{nf}$ as

$$\begin{aligned} \dot{A}_{nf} &= \varepsilon G_{av}(A,\varphi) \\ \dot{\varphi}_{nf} &= \varepsilon H_{av}(A,\varphi) \end{aligned} \qquad (6.10)$$

Further carrying out the averaging process over a cycle one gets, the non-'fluctuating' - non-'oscillatory' amplitude and phase terms as

$$\begin{aligned} \dot{A}_{nf-no} &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \varepsilon G_{av}[A,\varphi]dt \\ &= -[(F_0/2m\omega)\sin(\varphi-0)+(\omega_n\xi A)] \\ \dot{\varphi}_{nf-no} &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \varepsilon H_{av}[A,\varphi]dt \\ &= [-(\omega/2)-(F_o/2m\omega A)\cos(\varphi-0)+(\omega_n^2/2\omega)+(3\omega_n^2\lambda\ A^2/8\omega)] \end{aligned} \qquad (6.11)$$

Considering equations (6.7), (6.8) and (6.9) and putting the 'fluctuating' terms back into the above expressions, the non-'oscillating' approximations for the amplitude and phase are

$$\begin{aligned} \dot{A}_{no} &= -[(F_0 / 2m\omega)\sin(\varphi - \theta) + (\omega_n \xi A)] - \{\zeta(t)/\omega\}\sin(\omega t + \varphi) \\ \dot{\varphi}_{no} &= [-(\omega/2) - (F_o / 2m\omega A)\cos(\varphi - \theta) + (\omega_n^2 / 2\omega) + (3\omega_n^2 \lambda\ A^2 / 8\omega)] \\ &\quad -\{\zeta(t)/\omega A\}\cos(\omega t + \varphi) \end{aligned} \tag{6.12}$$

The equations (6.12) can be condensed as

$$\begin{aligned} \dot{A}_{no} &= \varepsilon G_1[A,\varphi] - \{\zeta(t)/\omega\}\sin(\omega t + \varphi) \\ \dot{\varphi}_{no} &= \varepsilon H_1[A,\varphi] - \{\zeta(t)/\omega A\}\cos(\omega t + \varphi) \\ &\text{with} \\ G_1[A,\varphi] &= -[(F_0\lambda / 2m\omega)\sin(\varphi - \theta) + \omega_n \lambda \xi A] \\ H_1[A,\varphi] &= [-(\omega\lambda/2) - (F_o\lambda/2m\omega A)\cos(\varphi - \theta) + (\omega_n^2\lambda/2\omega) + (3\omega_n^2\lambda^2 A^2 / 8\omega)] \end{aligned} \tag{6.13}$$

The 'truncated' equations (6.11), giving $\dot{A}_{nf-no}$ and $\dot{\varphi}_{nf-no}$ or the 'truncated' equations (6.12), giving $\dot{A}_{no}$ and $\dot{\varphi}_{no}$ can be taken as approximations of $\dot{A}$ and $\dot{\varphi}$. However, either approximation does not, adequately, reflect the influence of non- linearity in the system, for while the expression for $\dot{\varphi}_{nf-no}$ (or $\dot{\varphi}_{no}$) does involve the non-linearity parameter λ, the one for $\dot{A}_{nf-no}$ (or $\dot{A}_{no}$) does not. A higher order of approximation for $\dot{A}$ and $\dot{\varphi}$ is, therefore, essential to adequately represent the effects of the non-linearity on the statistical characteristics of the response.

6.4 SECOND ORDER AVERAGING

For a higher order approximation, instead of obtaining $\dot{A}_{no}$ and $\dot{\varphi}_{no}$ as in equation (6.12), the procedure of asymptotic method developed by Bogoliubov (Stratonovich, 1967) can be employed, whereby the non-'fluctuating' approximations $\dot{A}_{nf}$ and $\dot{\varphi}_{nf}$, of equation (6.10) are taken to have the form

$$\dot{A}_{nf} = \dot{A}^* + \varepsilon\dot{u}(A^*,\varphi^*)$$
$$\dot{\varphi}_{nf} = \dot{\varphi}^* + \varepsilon\dot{v}(A^*,\varphi^*) \tag{6.14}$$

where $\dot{A}^*$ and $\dot{\varphi}^*$ are expressed as

$$\dot{A}^* = \varepsilon G^*_{av}(A^*,\varphi^*) = \varepsilon G^*_1(A^*,\varphi^*) + \varepsilon^2 G^*_2(A^*,\varphi^*) + \ldots$$
$$\dot{\varphi}^* = \varepsilon H^*_{av}(A^*,\varphi^*) = \varepsilon H^*_1(A^*,\varphi^*) + \varepsilon^2 H^*_2(A^*,\varphi^*) + \ldots \tag{6.15}$$

Similarly the variations $\dot{u}$ and $\dot{v}$ are expressed in series form as

$$\dot{u}(A^*,\varphi^*) = \dot{u}_1(A^*,\varphi^*) + \varepsilon\dot{u}_2(A^*,\varphi^*) + \ldots$$
$$\dot{v}(A^*,\varphi^*) = \dot{v}_1(A^*,\varphi^*) + \varepsilon\dot{v}_2(A^*,\varphi^*) + \ldots \tag{6.16}$$

Equations (6.14), along with the series expansions of equations (6.15) and (6.16) are substituted into the stochastically averaged equations (6.10).

Noting that

$$\dot{u} = (\partial u/\partial A^*)\dot{A}^* + (\partial u/\partial\varphi^*)(\omega + \dot{\varphi}^*)$$
$$\dot{v} = (\partial v/\partial A^*)\dot{A}^* + (\partial v/\partial\varphi^*)(\omega + \dot{\varphi}^*) \tag{6.17}$$

the terms with identical power of ε are equated to obtain the equations governing the successive approximations. The equation governing the terms involving the first approximation (of order ε^1) is, thus, obtained as

$$G^*_1(A^*,\varphi^*) + \omega(\partial u_1/\partial\varphi^*) = G^*_{av}(A^*,\varphi^*)$$
$$H^*_1(A^*,\varphi^*) + \omega(\partial v_1/\partial\varphi^*) = H^*_{av}(A^*,\varphi^*) \tag{6.18}$$

(In the above equation, the term $G_{av}(A,\varphi)$, of equation (6.10) has been transformed to $G^*_{av}(A^*,\varphi^*)$)

The right hand side of equations (6.18) involve G^*_{av} and H^*_{av} , which contain both, the 'oscillatory' and the 'non-oscillatory' terms. The functions $u(A^*,\varphi^*)$ and $v(A^*,\varphi^*)$ are now chosen in such a way that $G^*_1(A^*,\varphi^*)$ and $H^*_1(A^*,\varphi^*)$ contain no oscillatory terms. Thus, terms involving $(\partial u/\partial\varphi^*)$ and

$(\partial v / \partial \varphi^*)$ are equated to the oscillatory parts of G^*_{av} and H^*_{av} , and $G_1^*(A^*,\varphi^*)$ and $H_1^*(A^*,\varphi^*)$ are equated to the non-oscillatory parts, to obtain

$$G_1^*(A^*,\varphi^*) = -\left[(F_0\lambda / 2m\omega)\sin(\varphi^* - \theta) + \omega_n \lambda \xi A^*\right]$$
$$H_1^*(A^*,\varphi^*) = \left[-(\omega\lambda/2) - (F_0\lambda / 2m\omega A^*)\cos(\varphi^* - \theta) + (\omega_n^2\lambda / 2\omega) + (3\omega_n^2\lambda^2 A^{*2} / 8\omega)\right] \quad (6.19)$$

and

$$\omega(\partial u_1(A^*,\varphi^*)/\partial\varphi^*) = \{-(\omega\lambda A^*/2) + (\omega_n^2\lambda A^*/2\omega) + (\omega_n^2\lambda^2 A^{*3}/4\omega)\}\sin 2(\omega t + \varphi^*)$$
$$+\{\omega_n\lambda\xi A^*\}\cos 2(\omega t + \varphi^*) - (F_0\lambda/2m\omega)\sin(2\omega t + \varphi^* + \theta)$$
$$+(\omega_n^2\lambda^2 A^3/8\omega)\sin 4(\omega t + \varphi)$$
$$\omega(\partial v_1(A^*,\varphi^*)/\partial\varphi^*) = -(\omega_n\lambda\xi)\sin 2(\omega t + \varphi^*) + \{-(\omega\lambda/2) + (\omega_n^2\lambda/2\omega)$$
$$+(\omega_n^2\lambda^2 A^{*2}/2\omega)\}\cos 2(\omega t + \varphi^*) - (F_0\lambda/2m\omega A^*)\cos(2\omega t + \varphi^* + \theta) \quad (6.20)$$
$$+(\omega_n^2\lambda^2 A^{*2}/8\omega)\cos 4(\omega t + \varphi^*)$$

Comparison of equations (6.19) and (6.13) reveals

$$G_1^*(A^*,\varphi^*) = G_1(A,\varphi)$$
$$H_1^*(A^*,\varphi^*) = H_1(A,\varphi) \quad (6.21)$$

Equations (6.20) give

$$u_1(A^*,\varphi^*) = \{(\lambda A^*/4) - (\omega_n^2\lambda A^*/4\omega^2) - (\omega_n^2\lambda^2 A^{*3}/8\omega^2)\}\cos 2(\omega t + \varphi^*)$$
$$+(\omega_n\lambda\xi A^*/2\omega)\sin 2(\omega t + \varphi^*) + (F_0\lambda/2m\omega^2)\cos(2\omega t + \varphi^* + \theta) \quad (6.22)$$
$$-(\omega_n^2\lambda^2 A^{*3}/32\omega^2)\cos 4(\omega t + \varphi^*)$$

$$v_1(A^*,\varphi^*) = (\omega_n\lambda\xi/2\omega)\cos 2(\omega t+\varphi^*) + \{-(\lambda/4)+(\omega_n^2\lambda/4\omega^2)$$
$$+(\omega_n^2\lambda^2 A^{*2}/4\omega^2)\}\sin 2(\omega t+\varphi^*) - (F_0\lambda/2m\omega^2 A^*)\sin(2\omega t+\varphi^*+\theta) \quad (6.23)$$
$$+(\omega_n^2\lambda^2 A^{*2}/32\omega^2)\sin 4(\omega t+\varphi^*)$$

For second approximation (of order ε^2), the governing equations are, similarly, obtained as

$$G_2^* + \omega(\partial u_2/\partial\varphi^*) = (\partial G_{av}^*/\partial A^*)u_1 + (\partial G_{av}^*/\partial\varphi^*)v_1 - (\partial u_1/\partial A^*)G_1^* - (\partial u_1/\partial\varphi^*)H_1^*$$
$$H_2^* + \omega(\partial v_2/\partial\varphi^*) = (\partial H_{av}^*/\partial A^*)u_1 + (\partial H_{av}^*/\partial\varphi^*)v_1 - (\partial v_1/\partial A^*)G_1^* - (\partial v_1/\partial\varphi^*)H_1^* \quad (6.24)$$

from which the non-oscillatory $G_2^*(A^*,\varphi^*)$ and $H_2^*(A^*,\varphi^*)$ are obtained as

$$G_2^*(A^*,\varphi^*) = (F_0\lambda/4m\omega^2)\{-(5\omega\lambda/4)+(5\omega_n^2\lambda/4\omega)+(A^{*2}\omega_n^2\lambda^2/\omega)\}\sin(\varphi^*-\theta)$$
$$+(5\omega_n\lambda\xi F_0/8m\omega^2)\cos(\varphi^*-\theta)$$

$$H_2^*(A^*,\varphi^*) = \{(3A^{*2}\omega_n^2\lambda^3/8\omega)-(3A^{*2}\omega_n^4\lambda^3/8\omega^3)-(51A^{*4}\omega_n^4\lambda^4/256\omega^3)-(\omega\lambda^2/8)$$
$$-(\omega_n^4\lambda^2/8\omega^3)\} - \{(F_0\lambda^2/4m\omega A^*)-(F_0\omega_n^2\lambda^2/4m\omega^3 A^*) \quad (6.25)$$
$$-(17A^*F_0\omega_n^2\lambda^3/32m\omega^3)\}\cos(\varphi^*-\theta) - (F_0\omega_n\lambda^2\xi/2m\omega^2 A^*)\sin(\varphi^*-\theta)$$

Substitution from equations (6.19) and (6.25) into equation (6.15) gives

$$\dot{A}^* = -\varepsilon[(F_0\lambda/2m\omega)\sin(\varphi^*-\theta)+\omega_n\lambda\xi A^*] + \varepsilon^2[(F_0\lambda/4m\omega^2)\{-(5\omega\lambda/4) \quad (6.26)$$
$$+(5\omega_n^2\lambda/4\omega)+(A^{*2}\omega_n^2\lambda^2/\omega)\}\sin(\varphi^*-\theta)+(5\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^*-\theta)]$$

$$\dot{\varphi}^* = \varepsilon[-(\omega\lambda/2)-(F_o\lambda/2m\omega A^*)\cos(\varphi^*-\theta)+(\omega_n^2\lambda/2\omega)+(3\omega_n^2\lambda^2 A^{*2}/8\omega)]$$
$$+\varepsilon^2[\{(3A^{*2}\omega_n^2\lambda^3/8\omega)-(3A^{*2}\omega_n^4\lambda^3/8\omega^3)-(51A^{*4}\omega_n^4\lambda^4/256\omega^3)-(\omega\lambda^2/8) \quad (6.27)$$
$$-(\omega_n^4\lambda^2/8\omega^3)\} - \{(F_0\lambda^2/4m\omega A^*)-(F_0\omega_n^2\lambda^2/4m\omega^3 A^*)$$
$$-(17A^*F_0\omega_n^2\lambda^3/32m\omega^3)\}\cos(\varphi^*-\theta)-(F_0\omega_n\lambda^2\xi/2m\omega^2 A^*)\sin(\varphi^*-\theta)]$$

Noting equation (6.14) and that the 'oscillatory' terms are confined to the variables u and v, the non-'fluctuating ' - non-'oscillatory' amplitude term A_{nf-no} becomes

$$\dot{A}_{nf-no} = \dot{A}^* \tag{6.28}$$

where $\dot{A}^*$ is given by equation (6.26).

Consideration of equations (6.28), (6.7) and (6.8) enables writing the non-'oscillatory' amplitude term, A_{no}, as

$$\begin{aligned}\dot{A}_{no} &= \dot{A}_{nf-no} - \varepsilon[(\zeta(t)\lambda/\omega)\sin(\omega t+\varphi^*)] \\ &= -\varepsilon[(F_0\lambda/2m\omega)\sin(\varphi^*-\theta)+\omega_n\lambda\xi A^*]+\varepsilon^2[(F_0\lambda/4m\omega^2)\{-(5\omega\lambda/4) \\ &\quad +(5\omega_n^2\lambda/4\omega)+(A^{*2}\omega_n^2\lambda^2/\omega)\}\sin(\varphi^*-\theta)+(5\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^*-\theta)] \\ &\quad -\varepsilon[(\zeta(t)\lambda/\omega)\sin(\omega t+\varphi^*)]\end{aligned} \tag{6.29}$$

Similarly

$$\begin{aligned}\dot{\varphi}_{no} &= \varepsilon[-(\omega\lambda/2)-(F_o\lambda/2m\omega A^*)\cos(\varphi^*-\theta)+(\omega_n^2\lambda/2\omega)+(3\omega_n^2\lambda^2A^{*2}/8\omega)] \\ &\quad +\varepsilon^2[\{(3A^{*2}\omega_n^2\lambda^3/8\omega)-(3A^{*2}\omega_n^4\lambda^3/8\omega^3)-(51A^{*4}\omega_n^4\lambda^4/256\omega^3)-(\omega\lambda^2/8) \\ &\quad -(\omega_n^4\lambda^2/8\omega^3)\}-\{(F_0\lambda^2/4m\omega A^*)-(F_0\omega_n^2\lambda^2/4m\omega^3A^*) \\ &\quad -(17A^*F_0\omega_n^2\lambda^3/32m\omega^3)\}\cos(\varphi^*-\theta)-(F_0\omega_n\lambda^2\xi/2m\omega^2A^*)\sin(\varphi^*-\theta)] \\ &\quad -\varepsilon[\{\zeta(t)\lambda/\omega A^*\}\cos(\omega t+\varphi^*)]\end{aligned} \tag{6.30}$$

The expressions in equations (6.29) and (6.30) are taken as approximations of the amplitude and phase terms, $\dot{A}$ and $\dot{\varphi}$, i.e.

$$\begin{aligned}\dot{A}^* &\approx \dot{A}_{no} \\ \dot{\varphi}^* &\approx \dot{\varphi}_{no}\end{aligned} \tag{6.31}$$

6.5 F-P-K EQUATION

The amplitude term, A^* and phase term, φ^* approximations in equation (6.31) are correlated with random excitation force $\zeta(t)$. However , since $\zeta(t)$ is assumed to be broad band random process, its correlation time is much smaller than the time constant characterising the rate of change of amplitude A^* and phase φ^*, which are slowly varying functions of time. It can be assumed that the values of $\zeta(t)$ are statistically independent of the values of A^* i.e. amplitude A^* can be approximated as a Markov process (Stratonovich, 1967; Khasminiskii, 1966; Papanicolaou and Kohler, 1974 and Lin, 1986). In addition amplitude changes much more rapidly than the phase, and hence the amplitude manages to establish an equilibrium amplitude distribution $p(A^*|\varphi^*)$ for every value of phase φ^*.

The modified FPK equation for the simplified equation (of the form as equations (6.29-6.31)) of the approximate Markov process is given as

$$\partial p(A^*,\varphi^*)/\partial t = -\partial/\partial A^*\{(a_1)p\} - \partial/\partial\varphi^*\{(b_1)\} + \varepsilon^2\partial^2/\partial A^{*2}\{(c_{11})p\} + \varepsilon^2\partial^2/\partial A^*\partial\varphi^*\{(d_{11})p\} + \varepsilon^2\partial^2/\partial\varphi^{*2}\{(e_{11})p\} \tag{6.32}$$

where modified drift coefficients are expressed as

$$\begin{aligned} a_1 &= [\varepsilon G_1^* + \varepsilon^2 G_2^* + \ldots] + \varepsilon^2 \int_{-\infty}^{0} K[\partial\dot{A}^*/\partial A^*, \dot{\varphi}_\tau^*]d\tau + \varepsilon^2 \int_{-\infty}^{0} K[\partial\dot{A}^*/\partial\varphi^*, \dot{\varphi}_\tau^*]d\tau \\ b_1 &= [\varepsilon H_1^* + \varepsilon^2 H_2^* + \ldots] + \varepsilon^2 \int_{-\infty}^{0} K[\partial\dot{\varphi}^*/\partial A^*, \dot{A}_\tau^*]d\tau + \varepsilon^2 \int_{-\infty}^{0} K[\partial\dot{\varphi}^*/\partial\varphi^*, \dot{\varphi}_\tau^*]d\tau \end{aligned} \tag{6.33}$$

and modified diffusion coefficients are

$$\begin{aligned} c_{11} &= \varepsilon^2 \int_{-\infty}^{0} K[\dot{A}^*, \dot{A}_\tau^*]d\tau \\ d_{11} &= \varepsilon^2 \int_{-\infty}^{0} \{K[\dot{A}^*, \dot{\varphi}_\tau^*] + K[\dot{\varphi}^*, \dot{A}_\tau^*]\}d\tau \\ e_{11} &= \varepsilon^2 \int_{-\infty}^{0} K[\dot{\varphi}^*, \dot{\varphi}_\tau^*]d\tau \end{aligned} \tag{6.34}$$

In equations (6.33) and (6.34) τ ($= (t_2 - t_1)$) is the time shift and $K[\dot{A}^*, \dot{\varphi}^*_\tau]$ is the correlation function. While computing the correlation function, the randomly fluctuating parts of the expressions $\dot{A}^*$ and $\dot{\varphi}^*$ (equations (6.29-6.31)) , i.e. the terms $-\{\zeta(t)\lambda / \omega\}\sin(\omega t + \varphi)$ and $-\{\zeta(t)\lambda / \omega A\}\cos(\omega t + \varphi)$, which contain the random force term $\zeta(t)$, need to be accounted for.

For the one dimensional approximate Markov process , A^* of equations (6.29) and (6.31), the drift coefficients are obtained as

$$\begin{aligned} a_1 = \{&-\varepsilon(F_0\lambda / 2m\omega)\sin(\varphi^* - \theta) - \varepsilon(\omega_n\lambda\xi A^*) + \varepsilon^2(F_0\lambda / 4m\omega^2)\{-(5\omega\lambda / 4) \\ &+(5\omega_n^2\lambda / 4\omega) + (A^{*2}\omega_n^2\lambda^2 / \omega)\}\sin(\varphi^* - \theta) + \varepsilon^2(5\omega_n\lambda^2\xi F_0 / 8m\omega^2)\cos(\varphi^* - \theta) \\ &+\varepsilon^2\{S(\zeta;\omega)\lambda^2 / 8\omega^2 A^*\}\} \end{aligned} \tag{6.35}$$

$$b_1 = 0$$

The diffusion coefficients are computed to be

$$\begin{aligned} c_{11} &= \varepsilon^2\{S(\zeta;\omega)\lambda^2 / 8\omega^2\} \\ d_{11} &= 0 \\ e_{11} &= 0 \end{aligned} \tag{6.36}$$

where the spectral density of noise $\zeta(t)$ at the frequency ω, defined as

$$S(\zeta(t),\omega) = 2\int_{-\infty}^{\infty} < \zeta(t_1)\zeta(t_2) > \cos\omega(t_2 - t_1)d(t_2 - t_1) \tag{6.37}$$

The Fokker Planck equation for amplitude A^*, from above equations , can now be written as

$$\begin{aligned} &-(\partial / \partial A^*)[\{-\varepsilon(F_0\lambda / 2m\omega)\sin(\varphi^* - \theta) - \varepsilon(\omega_n\lambda\xi A^*) + \varepsilon^2(F_0\lambda / 4m\omega^2)\{-(5\omega\lambda / 4) \\ &+(5\omega_n^2\lambda / 4\omega) + (A^{*2}\omega_n^2\lambda^2 / \omega)\}\sin(\varphi^* - \theta) + \varepsilon^2(5\omega_n\lambda^2\xi F_0 / 8m\omega^2)\cos(\varphi^* - \theta) \\ &+\varepsilon^2\{S(\zeta;\omega)\lambda^2 / 8\omega^2 A^*\}\}p] + \varepsilon^2\{S(\zeta;\omega)\lambda^2 / 8\omega^2\}(\partial^2 p / \partial A^{*2}) = \partial p(A^*) / \partial t \end{aligned} \tag{6.38}$$

6.6 RESPONSE

For a stationary case equation (6.38) reduces to

$$-(\partial/\partial A^*)[\{-\varepsilon(F_0\lambda/2m\omega)\sin(\varphi^*-\theta)-\varepsilon(\omega_n\lambda\xi A^*)+\varepsilon^2(F_0\lambda/4m\omega^2)\{-(5\omega\lambda/4)$$
$$+(5\omega_n^2\lambda/4\omega)+(A^{*2}\omega_n^2\lambda^2/\omega)\}\sin(\varphi^*-\theta)+\varepsilon^2(5\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^*-\theta) \qquad (6.39)$$
$$+\varepsilon^2\{S(\zeta;\omega)\lambda^2/8\omega^2A^*\}\}p]+\varepsilon^2\{S(\zeta;\omega)\lambda^2/8\omega^2\}(\partial^2p/\partial A^{*2})=0$$

The solution to the stationary Fokker-Planck equation, (6.39), turns out as

$$p(A^*)=cA^*\exp[-\{8\omega^2/\varepsilon^2\lambda^2S(\zeta,\omega)\}\{\varepsilon(A^*F_0\lambda/2m\omega)\sin(\varphi^*-\theta)$$
$$+\varepsilon(A^{*2}\omega_n\lambda\xi/2)-\varepsilon^2(F_0\lambda/4m\omega^2)\{-(5A^*\omega\lambda/4)+(5A^*\omega_n^2\lambda/4\omega) \qquad (6.40)$$
$$+(A^{*3}\omega_n^2\lambda^2/3\omega)\}\sin(\varphi^*-\theta)-\varepsilon^2(5A^*\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^*-\theta)\}]$$

6.7 PARAMETER ESTIMATION

The probability density function for any two values A_i^* and A_{i+1}^* ,of the amplitude (with $A_{i+1}^* \rangle A_i^*$), can be written from equation (6.40), as

$$p(A_i^*)=cA_i^*\exp[-\{8\omega^2/\varepsilon^2\lambda^2S(\zeta,\omega)\}\{\varepsilon(A_i^*F_0\lambda/2m\omega)\sin(\varphi^*-\theta)$$
$$+\varepsilon(A_i^{*2}\omega_n\lambda\xi/2)-\varepsilon^2(F_0\lambda/4m\omega^2)\{-(5A_i^*\omega\lambda/4)+(5A_i^*\omega_n^2\lambda/4\omega) \qquad (6.41)$$
$$+(A_i^{*3}\omega_n^2\lambda^2/3\omega)\}\sin(\varphi^*-\theta)-\varepsilon^2(5A_i^*\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^*-\theta)\}]$$

$$p(A_{i+1}^*)=cA_{i+1}^*\exp[-\{8\omega^2/\varepsilon^2\lambda^2S(\zeta,\omega)\}\{\varepsilon(A_{i+1}^*F_0\lambda/2m\omega)\sin(\varphi^*-\theta)$$
$$+\varepsilon(A_{i+1}^{*2}\omega_n\lambda\xi/2)-\varepsilon^2(F_0\lambda/4m\omega^2)\{-(5A_{i+1}^*\omega\lambda/4)+(5A_{i+1}^*\omega_n^2\lambda/4\omega) \qquad (6.42)$$
$$+(A_{i+1}^{*3}\omega_n^2\lambda^2/3\omega)\}\sin(\varphi^*-\theta)-\varepsilon^2(5A_{i+1}^*\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^*-\theta)\}]$$

Defining, $\Delta A_i^* = (A_{i+1}^* - A_i^*)$, for small ΔA_i^* , one can write, from equations (6.41) and (6.42)

$$
\begin{aligned}
[p(A_{i+1}^*)/p(A_i^*)] = (A_{i+1}^*/A_i^*)\exp[-\{8\omega^2/\varepsilon^2\lambda^2 S(\zeta,\omega)\}\{\varepsilon(\Delta A_i^* F_0\lambda/2m\omega)\sin(\varphi^* - \theta) \\
+\varepsilon(A_i^*\Delta A_i^*\omega_n\lambda\xi) - \varepsilon^2(F_0\lambda/4m\omega^2)\{-(5\Delta A_i^*\omega\lambda/4) \\
+(5\Delta A_i^*\omega_n^2\lambda/4\omega) + (A_i^{*2}\Delta A_i^*\omega_n^2\lambda^2/\omega)\}\sin(\varphi^* - \theta) \\
-\varepsilon^2(5\Delta A_i^*\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^* - \theta)\}]
\end{aligned}
\tag{6.43}
$$

For N amplitude values, $A_1^*, A_2^*, \ldots, A_N^*$, equation (6.43) is expressed as set of $(N-1)$ linear simultaneous algebraic equations,

$$
\begin{aligned}
&[(1/8\Delta A_i^*\omega^2)\ln\{A_{i+1}^* p(A_i^*)/A_i^* p(A_{i+1}^*)\}]\{S(\zeta,\omega)/\omega_n\xi F_0\cos\theta\} \\
&+[13/16m\omega]\{\sin(\varphi^* - \theta)/\omega_n\xi\cos\theta\} - [5/16m\omega^3]\{\omega_n\sin(\varphi^* - \theta)/\xi\cos\theta\} \\
&-[A_i^{*2}/4m\omega^3]\{\omega_n\lambda\sin(\varphi^* - \theta)/\xi\cos\theta\} + \left[A_i^*\right]\{1/F_0\cos\theta\} + [5\sin\varphi^*/8m\omega^2]\{\tan\theta\} \\
&= -[5\cos\varphi^*/8m\omega^2] \qquad\qquad i = 1,2,\ldots,(N-1)
\end{aligned}
\tag{6.44}
$$

Equations (6.44) are to estimate the parameters ω_n , λ, F_0 , ξ, $S(\zeta,\omega)$ and θ., using the Least Squares procedure.

The amplitude displacement and velocity data (x and $\dot{x}$) is obtained experimentally and using equation (6.6), the amplitude A , and phase φ are computed along with the probability function, $p(A)$, to be fed into equation (6.44) for parameter estimation. However, equation (6.44) involves A^*, φ^* and $p(A^*)$ and as an initial approximation the experimentally obtained A, φ and $p(A)$ are taken as A^*, φ^* and $p(A^*)$ respectively.

6.8 EXPERIMENTATION

The procedure described above is illustrated on the laboratory rotor. A disc of mass m= 0.41 Kgs. is centrally mounted on the shaft. The shaft, as in the previous illustrations, is mounted in the same bearings at its ends. Accelerometers are mounted on one of the bearing caps to sense the vibration signals. A reference

signal is picked up from the shaft by a non-contact eddy current proximity probe. The rotor is dynamically balanced and then a known unbalance mass is attached at the disc. The rotor configuration is same as in Chapter 3. The shaft is rotated at a particular speed and the signals are picked up. The experiment is repeated for different set of known unbalance masses and for different speeds.

Experimentally obtained displacement and velocity signals along with the corresponding reference signal are shown, for a rotor speed of 1800 rpm, in Figure 6.2.

6.8.1 Initial Approximation

The amplitude A and phase φ signals computed from these measured data are shown in Figures 6.3 and 6.4. The probability density function, $p(A)$, of the amplitude is shown in Figure 6.5. The bearing parameters estimated, for initial approximation of using experimentally obtained $A(t)$ as $A^*(t)$ and $\varphi(t)$ as $\varphi^*(t)$ in equations (6.44) are given in Table 6.1.

Table 6.1 **Experimentally estimated parameters (after initial approximation)**

Speed	Unbalance	Parameters estimated				
ω rpm	m e gm-cm (at $\theta = 0^0$)	φ degrees	θ degrees	m e gm-cm	ξ	k(x) N/mm
1800	10.5	0.7	0.1	16.80	0.062	$1.21 \times 10^4 - 0.58 \times 10^{10} x^2$
-	17.5	1.4	0.2	22.73	0.071	$1.23 \times 10^4 - 0.83 \times 10^{10} x^2$
-	24.5	6.9	0.1	30.64	0.022	$1.39 \times 10^4 - 1.06 \times 10^{10} x^2$
1400	10.5	3.0	0.3	17.24	0.083	$0.86 \times 10^4 - 0.84 \times 10^{10} x^2$
-	17.5	3.3	0.5	23.45	0.082	$0.79 \times 10^4 - 0.70 \times 10^{10} x^2$
	24.5	4.6	0.2	16.56	0.042	$1.12 \times 10^4 - 0.63 \times 10^{10} x^2$

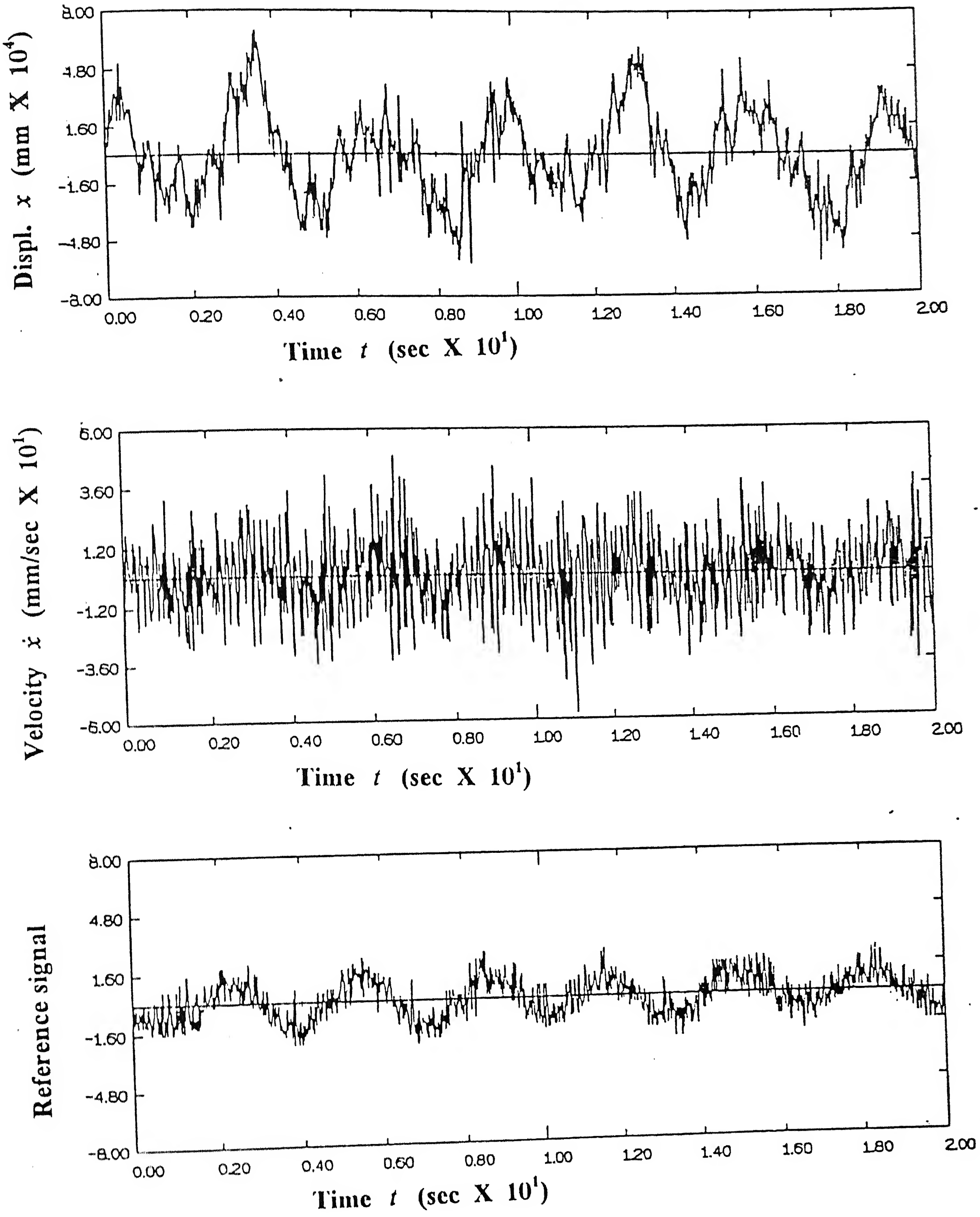


Figure 6.2 **Displacement, velocity and reference signals at 1800 rpm**

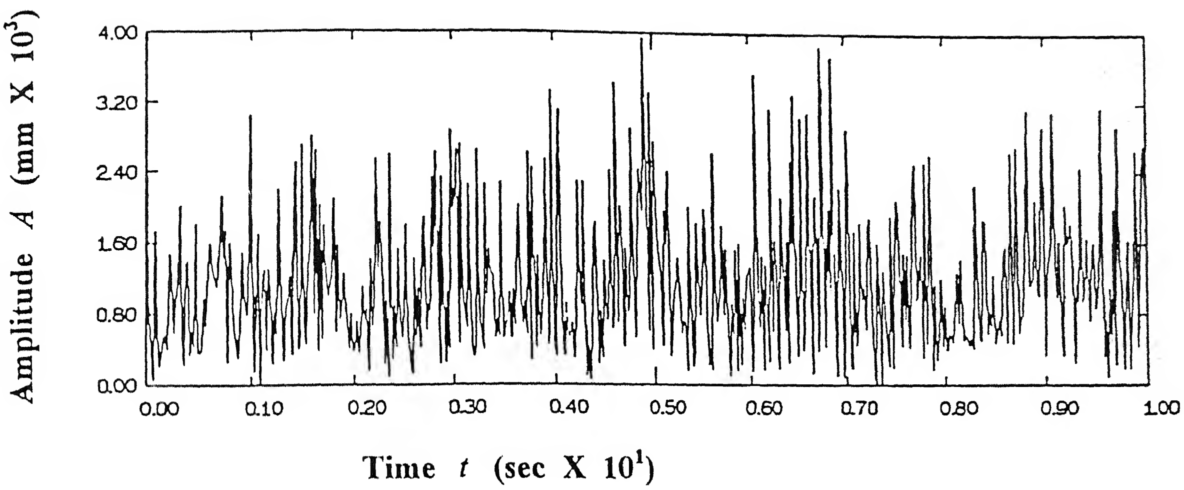


Figure 6.3 **Amplitude variation of the measured response**

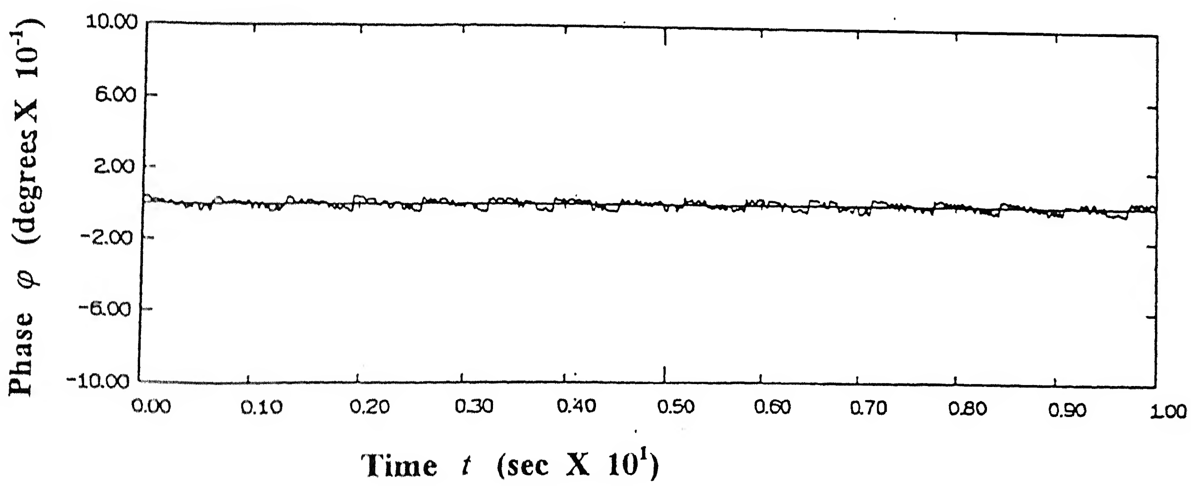


Figure 6.4 **Phase variation of the measured response**

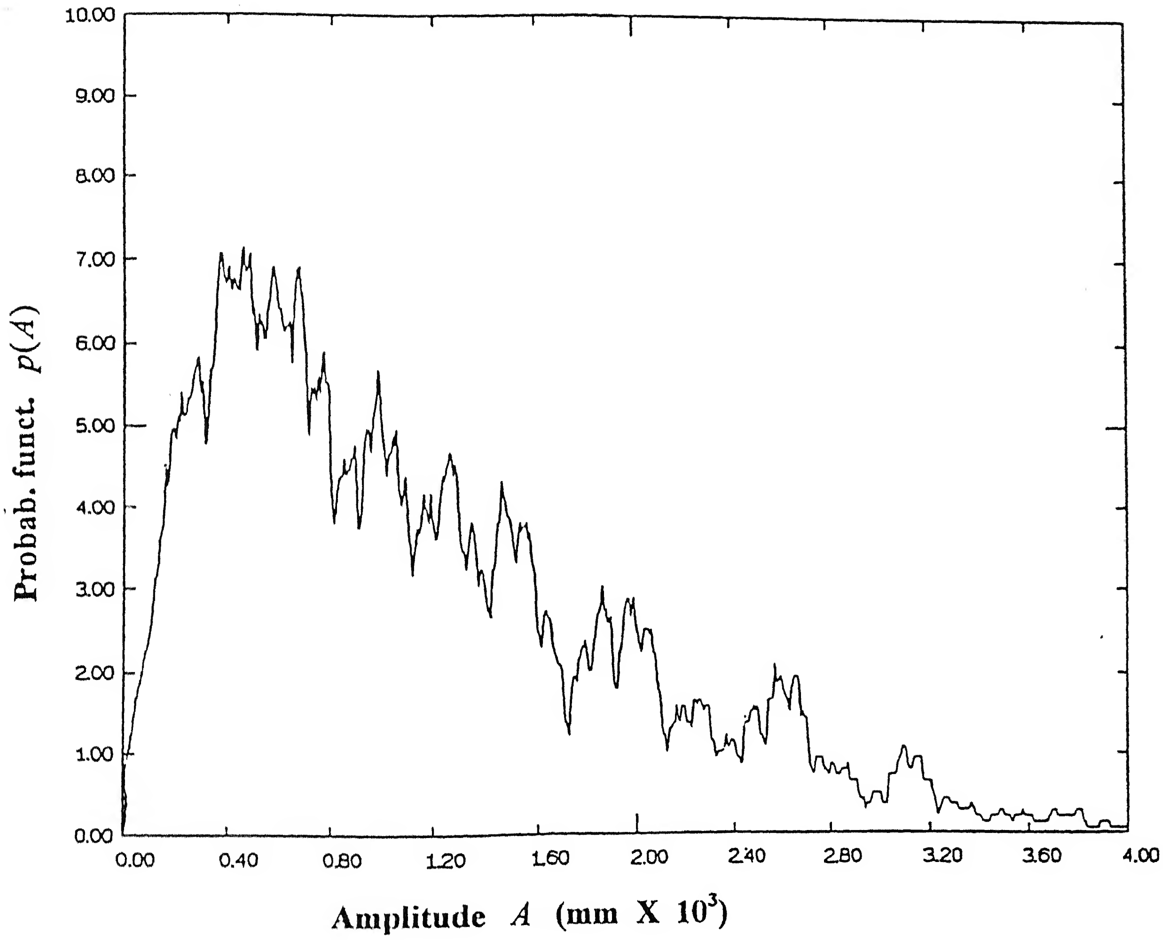


Figure 6.5 Probability distribution of the response amplitude

6.8.2 Iterative Scheme

For a more accurate estimation of estimated parameters an iterative scheme is devised. The parameters estimated from the above initial approximation are substituted in equations (6.22, 6.23), along with the initial assumption of taking the experimental A and φ to be A^* and φ^* respectively, to compute the variations u and v. These values of u and v and the experimental A and φ are employed in equation (6.14) to get a new approximations for A^* and φ^*. The new approximations, A^* and φ^*, are now employed in equation (6.44) for a fresh parameter estimation and thus, the iterative cycle can be continued. The final set of parameters estimated, after such iteration is given in Table 6.2.

Table 6.2 Experimentally estimated parameters (after iteration)

Speed	Unbalance	Parameters estimated				
ω rpm	m e gm-cm (at $\theta = 0^0$)	φ degrees	θ degrees	m e gm-cm	ξ	k(x) N/mm
1800	10.5	2.9	0.3	14.23	0.023	$1.31 \times 10^4 - 0.92 \times 10^{10} x^2$
-	17.5	1.6	0.1	20.32	0.036	$1.41 \times 10^4 - 1.10 \times 10^{10} x^2$
-	24.5	3.2	0.2	27.32	0.032	$1.60 \times 10^4 - 1.28 \times 10^{10} x^2$
1400	10.5	1.7	0.2	15.43	0.063	$0.94 \times 10^4 - 0.94 \times 10^{10} x^2$
-	17.5	3.2	0.4	21.40	0.063	$0.93 \times 10^4 - 1.01 \times 10^{10} x^2$
-	24.5	1.8	0.1	23.56	0.037	$1.37 \times 10^4 - 0.72 \times 10^{10} x^2$

(Refer to Appendix C for Statistical Error Table)

6.9 VALIDATION

The closeness, between the known unbalance introduced in the rotor and its angular location and the corresponding experimentally estimated values, is a measure of the validity of the procedure developed. The estimated damping ratios also appear physically reasonable. The bearing stiffness parameters obtained in the previous Chapter 3, employing the analytical formulations of Ragulskis et al. (1974) and Harris (1984), are reproduced in Table 6.3, for easy reference.

Table 6.3 Theoretical (Ragulskis et al., 1974 and Harris, 1984) bearing stiffness parameters

Preload (mm)	Theoretical Stiffness (Radial) (N/mm)
0.0002	$1.20 \times 10^4 - 4.01 \times 10^{10}\ x^2$
0.0003	$1.47 \times 10^4 - 2.18 \times 10^{10}\ x^2$
0.0004	$1.69 \times 10^4 - 1.42 \times 10^{10}\ x^2$
0.0005	$1.89 \times 10^4 - 1.02 \times 10^{10}\ x^2$
0.0006	$2.08 \times 10 - 6.09 \times 10^9\ x^2$

The estimated bearing stiffnesses of Table 6.3 can be seen to show good resemblance to theoretically possible values of Table 6.3.

6.10 REMARKS

The analysis presented underlines that stochastic averaging techniques can be usefully employed to address the inverse problem of parameter estimation in a rotor system governed by a nonlinear equation having both, harmonic and random forms of excitation. Also, along with the bearing stiffness parameters, estimates of the unknown unbalance of the rotor, its angular location and the damping ratio are also obtained, as by-products of the procedure developed.

CHAPTER 7

CONCLUSION

The present study has concerned itself with the inverse problem of parameter estimation in nonlinear rotor bearing systems from the vibration signals measurable at the bearing caps. Methods in statistical dynamics have provided the framework of analysis. Nonlinear rotor bearing system have been modeled as Markovian processes or as approximate Markovian processes after stochastic averaging to render standard form equations, which were solved to obtain the response statistics.

The following rotor systems were considered

(i) Balanced Rigid Rotor

(ii) Balanced Rigid Rotor With Flexible Disc

(iii) Balanced Multi-Disc Flexible Rotors

(iv) Unbalanced Rigid Rotor.

The parameter estimation procedures, were illustrated on a laboratory rotor model and the results were validated through comparison with those from available analytical guidelines. A Monte-Carlo simulation was also done to check the accuracies of the assumptions and the approximations involved.

In the first three cases, balanced rotors (or rotors with negligible unbalance) were considered and estimates of the nonlinear stiffness parameters of the bearings were obtained along with the estimates of linear parameters. The appropriateness of statistical methods and good estimates of the parameters, provided the motivation for taking up the case of a rotor with an unknown small, but not negligible unbalance (If the unbalance is noticeably high, then the priority is to balance the rotor). The statistical averaging technique and subsequent iterative refinements, adopted for this case, yield not only good estimates for the linear and nonlinear stiffness parameters of the bearings, but simultaneously also provide good approximations of the magnitude of the unknown unbalance as well as its location with respect to a reference point on the shaft. Reasonable values are obtained for the damping ratio too, as by-products. However these damping ratios need to be validated through an independent set.

These aspects highlight the suitability of the procedure, for usage as an on-line tool for parameter estimation, in contrast to other available techniques which either need the machine to be stopped to give it a known excitation or others which provide analytically possible ranges of stiffness of the bearing treated in isolation of the shaft. The possibility of monitoring the nonlinear part of the bearing stiffness for incipient bearing faults and its wear and tear needs to be explored further.

The unbalanced rotor analysis has been restricted to the case of a rigid rotor in flexible bearings, which could be treated as a single-degree-of-freedom system. The analysis needs to be extended to multi-degree-of freedom systems, by considering rotors on flexible shafts carrying more than one disc. If such an extension of the present scheme can be made, then along with the stiffness estimates, the possibility of obtaining approximations for the unbalances on the various discs (as obtained in the present study for a rigid rotor) need to be investigated.

Further, the problems have been formulated by taking the equivalent form linear viscous damping. The role of other forms damping in the analysis needs exploration. The present analysis, based on processing random vibrations, is restricted to rotors supported in rolling element bearings. The suitability of the analysis, for rotors in fluid film bearings needs to be investigated.

Instability phenomenon in rotors experiencing random excitations, and the influences of bearing stiffness nonlinearity have not been investigated in the present study. However, the present study, and its possible future extensions, can facilitate such investigations, by providing estimates of the parameters which form the causes of rotor instabilities.

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APPENDIX A

INSTRUMENTATION

A.1 SPECIFICATIONS OF INSTRUMENT:

1. Accelerometer (Piezo-electric):

Make - Bruel & Kjaer, Denmark	Type - 4370
Charge Sensitivity - 10.0 pC/ms^{-2}	Frequency Range - 0.1 Hz. ~ 5.0 kHz.
Construction - Delta Shear	Maximum Acceleration - 0.6 km/sec^2

2. Proximity Probe:

Make - National Aeronautical Laboratory, India	Type - Eddy current
Maximum Amplitude Allowable -	2 mm peak to peak.
Frequency Range -	50 Hz to 1 kHz

3. Digital Tachometer:

Make - Teclock Corporation, Japan	Range - 0 to 10 000 rpm

4. Phase Meter:

Make - Bruel & Kjaer, Denmark	Type - 2977
Inputs - Two BNC Sockets -	A - Reference input
	B - Unknown Phase input
Phase Accuracy -	$\pm 0.1^0$ absolute error.

5. Tunable Band Pass Filter:

Make - Bruel & Kjaer, Denmark	Type - 1621
Input - BNC Socket	Frequency Range - 0.2 Hz. ~ 20 kHz.
Selective Bandwidth -	3% or 23% (1/3 Octave)

6. Charge Amplifier:

Make - Bruel & Kjaer, Denmark	Type - 2635
Calibrated Output ratings -	Selectable in 10 dB steps,
	0.1 mV to 1 V/ms^{-2} (Acceleration),
	10 mV to 100 V/ms^{-1} (Velocity),
	0.1 mV to 10 V/mm (Displacement)
Frequency Range -	Switchable
	0.2 or 2 Hz. to 100 kHz.(Acceleration),
	1 or 10 Hz to 10 kHz (Velocity)
	1 or 10 Hz to 1 kHz (Displacement)

7. Photoelectric tachometer Probe:

Make - Bruel & Kjaer, Denmark	Type - MM 0012
Transducer Type -	Combined infra-red light source and photo-sensor, fitted with infra-red filter.
Minimum Sensitive -	100 mV at 10 mm distance from a flat white card.
Response Time -	200 μs minimum for full output. Equivalent to a 10 mm long reflecting surface passing at 50 m/sec
Connector -	BNC with double shield (BNT)

8. Digital Storage Oscilloscope (DSO):

Make -Larsen & Toubro and Gould Electronics	Type - 4072
Input - 2 Channel BNC Socket	Frequency Range - 0 ~ 100 MHz (DC)
Screen Update Rate - 30 Traces per sec.	Maxi. Sampling Rate - 400 M samples/sec
Interface Bus - IEEE-488	

9. GPIB-PC:

Type - IEEE-488	Connector - Shielded 24 pin conductor cable
Maximum Frequency Range - 100 MHz.	Connected to - PC/AT
Transfer Rate -	5K bytes/sec in binary mode

APPENDIX B

EIGEN VALUES AND EIGEN VECTORS OF FLEXIBLE ROTOR-BEARING SYSTEMS WITH SIGLE DISC

B.1 EIGEN VALUES:

$$p_1^2 = 0$$

$$p_{2,3}^2 = \frac{12EI}{m_3 L^3}\frac{m_1 m_3 + 2m_1 m_2 + m_2 m_3 \mp \sqrt{-4m_1 m_2 m_3 (m_1 + m_2 + m_3) + (m_1 m_3 + 2m_1 m_2 + m_2 m_3)^2}}{2m_1 m_2 m_3}$$

B.2 EIGEN VECTORS:

$$U = \begin{bmatrix} 1 & u_{31} & u_{21} \\ 1 & u_{32} & u_{22} \\ 1 & 1 & 1 \end{bmatrix}$$

with

$$u_{21} = -1 + \frac{1}{4m_1^2 m_2 m_3}\left[\left\{m_1 m_3 - 2m_1 m_2 - m_2 m_3 + \sqrt{m_1^2 m_3^2 - 2m_1 m_2 m_3^2 + 4m_1^2 m_2^2 + m_2^2 m_3^2}\right\}\right.$$
$$\left.\left\{-m_1 m_3 + 2m_1 m_2 - m_2 m_3 + \sqrt{m_1^2 m_3^2 - 2m_1 m_2 m_3^2 + 4m_1^2 m_2^2 + m_2^2 m_3^2}\right\}\right]$$

$$u_{22} = \frac{\left\{m_1 m_3 - 2m_1 m_2 - m_2 m_3 + \sqrt{m_1^2 m_3^2 - 2m_1 m_2 m_3^2 + 4m_1^2 m_2^2 + m_2^2 m_3^2}\right\}}{2m_1 m_3}$$

$$u_{22} = -1 + \frac{1}{4m_1^2 m_2 m_3}\left[\left\{m_1 m_3 - 2m_1 m_2 - m_2 m_3 - \sqrt{m_1^2 m_3^2 - 2m_1 m_2 m_3^2 + 4m_1^2 m_2^2 + m_2^2 m_3^2}\right\}\right.$$
$$\left.\left\{-m_1 m_3 + 2m_1 m_2 - m_2 m_3 - \sqrt{m_1^2 m_3^2 - 2m_1 m_2 m_3^2 + 4m_1^2 m_2^2 + m_2^2 m_3^2}\right\}\right]$$

$$u_{32} = \frac{\left\{m_1 m_3 - 2m_1 m_2 - m_2 m_3 + \sqrt{m_1^2 m_3^2 - 2m_1 m_2 m_3^2 + 4m_1^2 m_2^2 + m_2^2 m_3^2}\right\}}{2m_1 m_3}$$

APPENDIX C

STATISTICAL ESTIMATION ERRORS

The statistical errors have been obtained by processing ensembles of data through the parameter estimation algorithms. The estimated values of the parameters for a typical ensembles of five signals each have been presented here in tabular form. Table C.1 concerns the rigid rotor configuration of Chapter 3; Table C.2, C.3 and C.4 refer to the rotor configurations of Chapters 4, 5 and 6 respectively.

A typical ensembles of displacement and velocity signals have been shown in Figs C.1 and C.2. The estimated parameters presented in Table C.1 refer to these figures.

Table C.1 Statistical Errors in Single Disc Rigid Rotor Case (Chapter 3)

	Parameters			
	ω_n^2 (rads/sec)^2	Variation	λ (mm^{-2})	Variation
Vertical	5.42×10^7 5.32×10^7 5.37×10^7 5.36×10^7 5.37×10^7	1.85%	-1.27×10^6 -1.23×10^6 -1.24×10^6 -1.25×10^6 -1.25×10^6	3.15%
Horizontal	3.21×10^7 3.14×10^7 3.11×10^7 3.18×10^7 3.16×10^7	3.11%	-1.29×10^6 -1.24×10^6 -1.25×10^6 -1.24×10^6 -1.27×10^6	3.88%

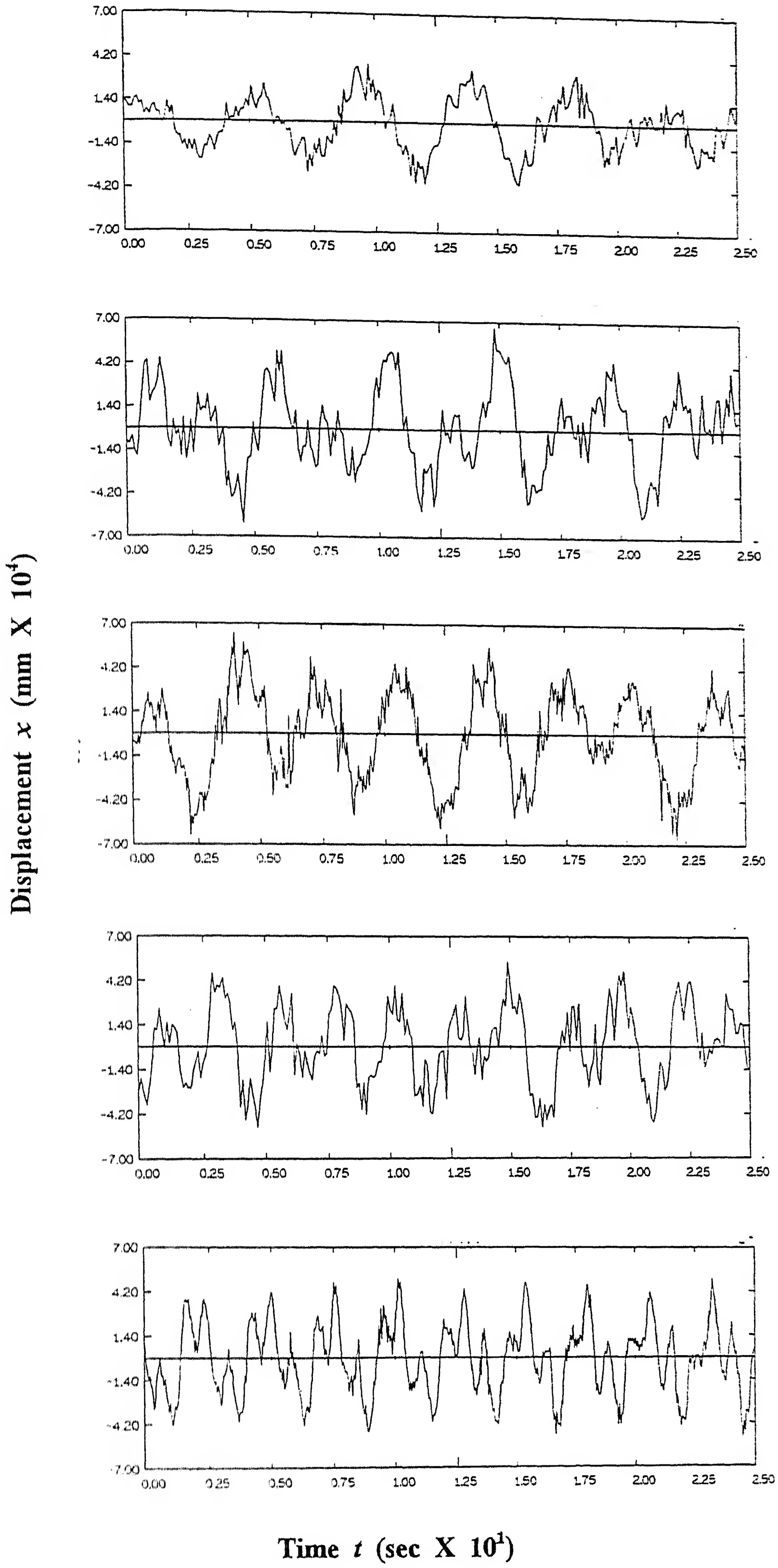

Figure C.1 Typical Ensemble of Displacement Signals for the Rigid Rotor Case (Chapter 3)

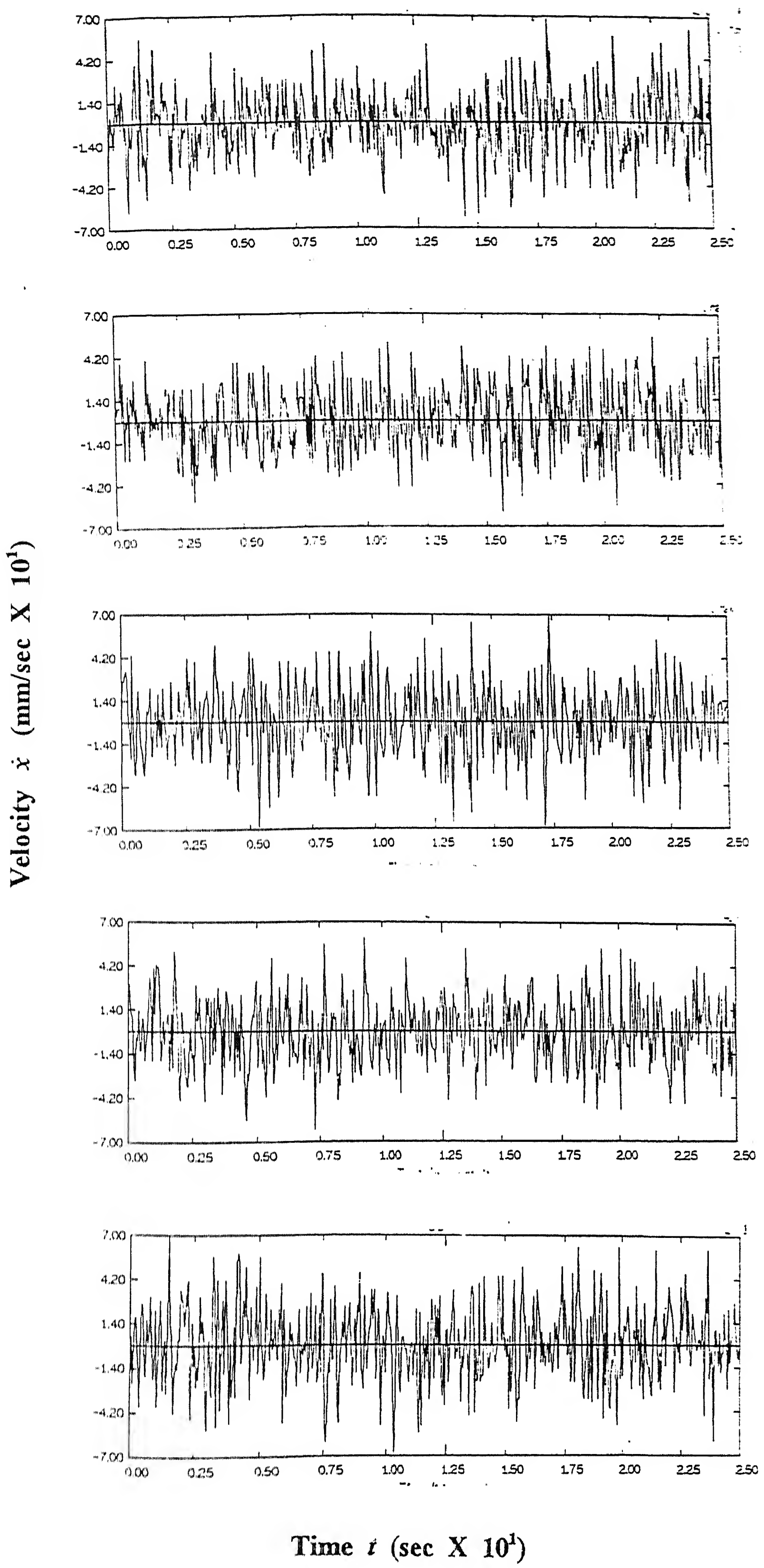

Figure C.2 Typical Ensemble of Velocity Signals for the Rigid Rotor Case (Chapter 3)

Table C.2 Statistical Errors in Flexible Single Disc Rotor Case (Chapter 4)

Parameters	Vertical	Variation	Horizontal	Variation
$\omega_{n_1}^2$ (rads/sec)^2	2.13×10^7 2.21×10^7 2.14×10^7 2.17×10^7 2.19×10^7	3.62%	2.12×10^7 2.15×10^7 2.17×10^7 2.16×10^7 2.15×10^7	2.30%
λ_1 (mm)$^{-2}$	-1.25×10^6 -1.24×10^6 -1.21×10^6 -1.25×10^6 -1.23×10^6	3.20%	-1.41×10^6 -1.44×10^6 -1.42×10^6 -1.43×10^6 -1.42×10^6	2.08%
$\omega_{n_2}^2$ (rads/sec)^2	1.98×10^7 1.92×10^7 1.96×10^7 1.94×10^7 1.97×10^7	3.03%	2.22×10^7 2.24×10^7 2.25×10^7 2.21×10^7 2.24×10^7	1.78%
λ_2 (mm)$^{-2}$	-1.15×10^6 -1.16×10^6 -1.13×10^6 -1.17×10^6 -1.15×10^6	3.42%	-1.36×10^6 -1.32×10^6 -1.35×10^6 -1.34×10^6 -1.32×10^6	2.94%
$\sqrt{\mu_1 \mu_2}$	2.10 2.06 2.12 2.09 2.11	2.83%	2.34 2.30 2.35 2.34 2.33	2.13%

Table C.3 Statistical Errors in Flexible Multi Disc Rotor Case (Chapter 5)

Parameters	Vertical	Variation	Horizontal	Variation
k_{L_1} (N/mm)	1.04×10^4 1.01×10^4 1.05×10^4 1.03×10^4 1.05×10^4	3.81%	0.87×10^4 0.86×10^4 0.84×10^4 0.85×10^4 0.86×10^4	3.45%
k_{NL_1} (N/mm^3)	-5.10×10^{10} -5.08×10^{10} -5.06×10^{10} -5.09×10^{10} -5.06×10^{10}	0.78%	-3.50×10^{10} -3.57×10^{10} -3.55×10^{10} -3.57×10^{10} -3.52×10^{10}	1.96%
k_{L_2} (N/mm)	1.04×10^4 1.03×10^4 1.06×10^4 1.04×10^4 1.05×10^4	2.83%	0.86×10^4 0.87×10^4 0.84×10^4 0.85×10^4 0.87×10^4	3.45%
k_{NL_2} (N/mm^3)	-3.62×10^{10} -3.58×10^{10} -3.61×10^{10} 3.62×10^{10} 3.59×10^{10}	1.10%	-2.19×10^{10} -2.21×10^{10} -2.22×10^{10} -2.19×10^{10} -2.20×10^{10}	1.35%
$\sqrt{m_1 m_2}$ (Kg)	0.21 0.20 0.21 0.21 0.20	4.76%	0.20 0.21 0.20 0.21 0.20	4.76%

Table C.4 Statistical Errors in Unbalanced Rotor Case (Chapter 6)

Speed	Unbalance	Parameters estimated				
ω rpm	m e gm-cm (at $\theta = 0^0$)	φ degrees	θ degrees	m e gm-cm	ξ	k(x) N/mm
1800		2.9	0.3	14.23	0.023	$1.31 \times 10^4 - 0.92 \times 10^{10} x^2$
		2.8	0.3	14.42	0.022	$1.32 \times 10^4 - 0.91 \times 10^{10} x^2$
	10.5	2.9	0.3	14.32	0.022	$1.34 \times 10^4 - 0.93 \times 10^{10} x^2$
Variation		2.8	0.3	14.30	0.023	$1.33 \times 10^4 - 0.91 \times 10^{10} x^2$
		2.9	0.3	14.36	0.022	$1.32 \times 10^4 - 0.92 \times 10^{10} x^2$
		(3.45%)	(0.00%)	(1.32%)	(4.35%)	(2.24%) (2.15%)
-		1.6	0.1	20.32	0.036	$1.41 \times 10^4 - 1.10 \times 10^{10} x^2$
	17.5	1.6	0.1	20.12	0.036	$1.44 \times 10^4 - 1.09 \times 10^{10} x^2$
		1.6	0.1	20.41	0.035	$1.45 \times 10^4 - 1.07 \times 10^{10} x^2$
Variation		1.6	0.1	20.24	0.036	$1.43 \times 10^4 - 1.08 \times 10^{10} x^2$
		1.6	0.1	20.31	0.035	$1.45 \times 10^4 - 1.09 \times 10^{10} x^2$
		(0.00%)	(0.00%)	(1.42%)	(2.78%)	(2.76%) (2.73%)
-		3.2	0.2	27.32	0.032	$1.60 \times 10^4 - 1.28 \times 10^{10} x^2$
		3.3	0.2	27.41	0.031	$1.62 \times 10^4 - 1.27 \times 10^{10} x^2$
	24.5	3.2	0.2	27.36	0.031	$1.63 \times 10^4 - 1.25 \times 10^{10} x^2$
Variation		3.2	0.2	27.38	0.032	$1.61 \times 10^4 - 1.26 \times 10^{10} x^2$
		3.2	0.2	27.40	0.031	$1.63 \times 10^4 - 1.27 \times 10^{10} x^2$
		(3.03%)	(0.00%)	(0.33%)	(3.13%)	(1.84%) (2.34%)
1400		1.7	0.2	15.43	0.063	$0.94 \times 10^4 - 0.94 \times 10^{10} x^2$
		1.7	0.2	15.23	0.061	$0.91 \times 10^4 - 0.92 \times 10^{10} x^2$
	10.5	1.7	0.2	15.34	0.064	$0.93 \times 10^4 - 0.93 \times 10^{10} x^2$
Variation		1.7	0.2	15.30	0.062	$0.94 \times 10^4 - 0.93 \times 10^{10} x^2$
		1.7	0.2	15.21	0.063	$0.93 \times 10^4 - 0.92 \times 10^{10} x^2$
		(0.00%)	(0.00%)	(1.30%)	(4.69%)	(3.19%) (2.13%)
-		3.2	0.4	21.40	0.063	$0.93 \times 10^4 - 1.01 \times 10^{10} x^2$
		3.1	0.4	21.32	0.062	$0.92 \times 10^4 - 1.02 \times 10^{10} x^2$
	17.5	3.2	0.4	21.37	0.063	$0.94 \times 10^4 - 1.04 \times 10^{10} x^2$
Variation		3.1	0.4	21.41	0.063	$0.93 \times 10^4 - 1.03 \times 10^{10} x^2$
		3.2	0.4	21.35	0.062	$0.94 \times 10^4 - 1.01 \times 10^{10} x^2$
		(3.13%)	(0.00%)	(0.42%)	(1.59%)	(2.13%) (2.88%)
-		1.8	0.1	23.56	0.037	$1.37 \times 10^4 - 0.72 \times 10^{10} x^2$
		1.8	0.1	23.42	0.036	$1.36 \times 10^4 - 0.71 \times 10^{10} x^2$
	24.5	1.8	0.1	23.61	0.037	$1.35 \times 10^4 - 0.73 \times 10^{10} x^2$
Variation		1.8	0.1	23.59	0.036	$1.36 \times 10^4 - 0.72 \times 10^{10} x^2$
		1.8	0.1	23.53	0.037	$1.37 \times 10^4 - 0.71 \times 10^{10} x^2$
		(0.00%)	(0.00%)	(0.80%)	(2.70%)	(1.46%) (2.74%)

LIST OF PUBLICATIONS FROM PRESENT STUDY

Tiwari, R. and Vyas, N. S., (1995), *Journal of Sound and Vibration* **187**(2), 229-239. Estimation of nonlinear stiffness parameters of rolling element bearings from random response of rotor bearing systems.

Tiwari, R. and Vyas, N. S. , (1995), (communicated for publication) *Journal of Sound and Vibration.* Nonlinear bearing stiffness parameter extraction from random response in flexible rotor bearing systems.

Tiwari, R. and Vyas, N. S., (1995), (communicated for publication) *Probabilistic Engineering Mechanics.* Stiffness estimation from random response in multi-mass rotor bearing systems.

Tiwari, R. and Vyas N. S., (1995), (communicated for publication) *Journal of Sound and Vibration.* Parameter estimation in unbalanced nonlinear rotor-bearing systems from random response.